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Modeling Statistic Distributions for Nonparametric Goodness-of-Fit Criteria for Testing Complex Hypotheses with Respect to the Inverse Gaussian Law[1]

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Abstract—We give percentage point tables and statistic distribution models for non-parametric goodness-of-fit criteria for testing complex hypothesis with respect to the inverse Gaussian law in case of using maximal likelihood estimates.

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1. INTRODUCTION. COMPUTER TECHNOLOGIES IN STUDYING STATISTICAL PATTERNS

The practice of using statistical analysis in various applications, including reliability problems, is rich with problem statements that do not conform to classical assumptions. A wide spectrum of statistical analysis methods is based on the assumption that measurement errors are distributed according to the normal law. In real life, the "normality" assumption, and often other assumptions, do not hold. Using classical methods of mathematical statistics in these situations may be incorrect.

Many classical results have an asymptotic nature, while in practice one usually deals with finite, often very limited, sample sizes. In these situations, applying asymptotic results is not always correct either.

The representation (registration) form of the data (measurements) often does not correspond to point samples considered in mathematical statistics textbooks. Real observations (samples) may be grouped, partially groups, censored, multiply censored, interval; this also restricts the use of classical methods and results.

In problems, for example, of reliability and lifetime of complex systems, quality control for high-reliability devices, survival analysis for complex chronic diseases and so on, researchers have been lately trying to use dynamic regression models, develop statistical methods for accelerated sampling for the analysis of complex industrial and medical experiments in dynamically changing environments. As in many other cases, wide application of these new approaches is held back, on one hand, by the development of mathematical apparatus, and, on the other hand, by the lack of software support that would help develop this apparatus and apply the developed approaches.

Finding fundamental statistical patterns in nonstandard application conditions is usually a hard problem. At the same time, analytic methods of studying these patterns (e.g., statistical properties of the estimates of criterion statistics' distributions) are exceedingly complex and do not allow,

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due to this complexity, to solve all of the problems. A possible solution is to widely use numerical methods related to computerized modeling of statistical patterns under conditions imitating the real measurement conditions and, then, constructing mathematical models approximating the found pattern. This approach obtains good results in situations when analytic methods alone fail. Therefore, computerized modeling and statistical pattern analysis methods have been getting into wider and wider use lately.

At present, there are many statistical analysis systems widely used in different applications of goodness-of-fit criteria. The CTI (The Computers in Teaching Initiative—an organization uniting British universities) database contains more than 100 packages implementing statistical analysis systems. In Russia, STATISTICA, SPSS, and SAS systems are most popular. Some software systems from this list are universal systems oriented towards a maximally wide spectrum of statistical analysis methods; others are tailored for a relatively narrow class of problems. The above-mentioned systems provide instruments for solving statistical analysis problems in various applications. However, these systems usually cannot serve as an instrument for studying patterns in mathematical statistics itself, for developing its mathematical apparatus.

Analyzing the latest trends shows that an increasing number of works uses, in studying the properties of estimates and statistics, numerical analysis and statistical modeling methods in order to support analytic conclusions. More and more often, computer technologies are used to develop applied mathematical statistics. In particular, with the help of the computerized approach and its development the authors have obtained a number of results useful for practice. For testing complex hypotheses, when a scalar or vector parameter of the probability distribution law is calculated from the same sample, nonparametric goodness-of-fit criterions of Kolmogorov, Cramér–Mises–Smirnov, and Anderson–Darling lose their distribution independency. For testing complex hypotheses, conditional distributions of criterion statistics depend on a number of factors.

Studies of the distributions of nonparametric goodness-of-fit criteria statistics for simple and complex goodness of fit hypotheses with a number of laws most often used in applications [1–3], constructed statistic distribution models for various complex hypotheses and percentile tables have been included in the developed standartization recommendations R 50.1.037-2002 [4]. At present, these results have been extended and made more precise [5–8].

We have constructed asymptotically optimal grouping tables for a wide enough range of distributions most often used in applications. Using asymptotically optimal grouping tables provides for the maximal power of criterion like χ^2 for close competing hypotheses [9]. We have studied the dependency between the power and the number of intervals and, for the first time, have shown that there exists an optimal number of intervals depending on sample size, precise alternatives, and the grouping method [10–13]. A part of these results was included in standartization recommendations R 50.1.033-2001 [14].

Information in various sources on the advantages in certain situations of a certain criterion is ambiguous and often contradictory. Estimates for the criteria's asymptotic power are hard to use due to bounded sample sizes that one deals with in practice. Studying the power is all the harder since there are no results on statistics distribution laws when competing hypotheses hold. Results on the estimation of goodness-of-fit criteria power of relatively close competing hypotheses, presented in [15–17], allow to order the criteria with respect to their power.

In [18, 19], we have shown that in certain cases, even for significant censoring the losses in Fischer information due to sample censoring are small. This lets us obtain good estimates for the law's parameters. With computerized modeling, we have studied distribution laws for the maximal likelihood estimates (MLE) for distribution parameters by censored observations for varying degrees of censoring and full sample sizes. We have shown that for bounded sample sizes MLE distributions turn out to be asymmetric, and the MLEs are biased.

In statistical modeling studies for classical statistics used for testing hypotheses on expectations and variances, we have shown that for testing hypotheses on expectations, using classical results remains correct for significant deviations of the observed law from the normal law [20]. This conclusion also holds for parametric criteria like the Student criterion, used for testing homogeneity hypotheses for two sample averages [21]. We have studied the stability and power of the Abbe criterion used for testing hypotheses on the lack of trend [22].

For statistics used in testing variance hypotheses, we have obtained percentile tables that can be used for observed laws described by the exponential distribution family [20]. For statistics used in Bartlett's and Cochran's criteria, we have we have obtained percentile tables that can be used for observed laws described by the exponential distribution family [23].

Percentile tables for statistics of Grubbs type for outlier testing of three maximal (three minimal) values simultaneously and the minimal and maximal value in a sample, simultaneously. Statistical modeling has allowed to study distributions of statistics for Grubbs criteria used for outlier rejection problems when the observed law differs from the normal law [24].

We have studied the power of the Smirnov and Lehmann–Rosenblatt homogeneity criteria for two samples. We have presented an adjustment for the Smirnov statistic that enhances convergence of the statistic's distribution to the limit law [25].

We have developed methods for modeling and studying distribution laws for arbitrary functions of random values and functions of systems of random values and for constructing approximate models for these laws [26].

We have studied statistic distributions and power of a number of criteria for deviation from the normal law. We have compared them with the power of goodness-of-fit criteria. We have shown disadvantages of several popular criteria [27, 28], in particular, a bias in Shapiro–Wilk and Epps–Pally criteria with respect to certain competing hypotheses.

We have developed a technique for modeling statistic distributions for multidimensional random values and studying statistic distributions for multidimensional random values [29].

It is certain that computer technologies for data and statistical pattern analysis present a powerful tool for developing and advancing applied mathematical statistics, including reliability, longevity, and survival analysis problems.

2. NONPARAMETRIC GOODNESS-OF-FIT CRITERIA FOR TESTING SIMPLE AND COMPLEX HYPOTHESES

Nonparametric goodness-of-fit criteria: Kolmogorov, Cramér–Mises–Smirnov ω^2, Anderson-Darling Ω^2,— for testing simple hypotheses H_0: $F(x) = F(x, \theta)$, where the theoretical distribution $F(x, \theta)$ is completely defined, do not depend on the law with which goodness-of-fit is tested.

In the Kolmogorov criterion, the following value is used as a distance between the empirical and theoretical laws:

$$D_n = \sup_{|n|<\infty} |F_n(x) - F(x, \theta)|,$$

where $F(x)$ is the empirical distribution function, n is the sample size. For $n \to \infty$, the distribution function of the $\sqrt{n}D_n$ statistic, if the tested hypothesis is true, uniformly converges to Kolmogorov's density function [30].

For testing hypotheses with Kolmogorov criterion, it is recommended to use the statistic with Bolshev correction [31, 32] in the form [33]

$$S_K = \frac{6nD_n + 1}{6\sqrt{n}}, \tag{1}$$

where

$$D_n = \max(D_n^+, D_n^-),$$
$$D_n^+ = \max_{1 \leqslant i \leqslant n} \left\{ \frac{i}{n} - F(x_i, \theta) \right\}, \quad D_n^- = \max_{1 \leqslant i \leqslant n} \left\{ F(x_i, \theta) - \frac{i-1}{n} \right\},$$

n is the sample size, $x_1, x_2, \ldots, x_n$ are sample elements ordered in ascending order. If the simple tested hypotheses holds, the statistic (1) is also distributed according to Kolmogorov's distribution [33] and converges significantly faster than the $\sqrt{n}D_n$ statistic.

The Cramér–Mises–Smirnov ω^2 criterion uses the following statistic [33]:

$$S_\omega = \frac{1}{12n} + \sum_{i=1}^{n} \left\{ F(x_i, \theta) - \frac{2i-1}{2n} \right\}^2, \tag{2}$$

and the Anderson–Darling Ω^2 criterion [34, 35] uses a statistic of the form

$$S_\Omega = -n - 2\sum_{i=1}^{n} \left\{ \frac{2i-1}{2n} \ln F(x_i, \theta) + \left(1 - \frac{2i-1}{2n}\right) \ln(1 - F(x_i, \theta)) \right\}. \tag{3}$$

For testing simple hypotheses, the statistic (2) has the $a1(S)$ distribution [33], and statistic (3) has the $a2(S)$ distribution [33].

For testing complex hypotheses $H_0 : F(x) \in \{F(x, \theta), \theta \in \Theta\}$, when an estimate of a scalar or vector distribution parameter is computed according to the same sample, nonparametric goodness-of-fit criteria: Kolmogorov, ω^2 Cramér–Mises–Smirnov, Anderson–Darling Ω^2, lose their distribution independence property. In this case, conditional statistic distributions $G(S \mid H_0)$ become dependent on a number of factors: on the form of the observed law $F(x, \theta)$ corresponding to the correct tested hypothesis H_0; on the type of the estimated parameter and the number of estimated parameters; in many cases (e.g., for families of gamma- and beta-distributions), on the precise value of the parameter; on the parameter estimation method.

Differences in limit distributions of the same statistic in testing simple and complex hypotheses are so significant that these differences certainly cannot be ignored. Despite this fact, the practice of applying nonparametric criteria abounds with incorrect use of classical results, which hold only for testing simple hypotheses, in cases when one actually tests complex hypotheses.

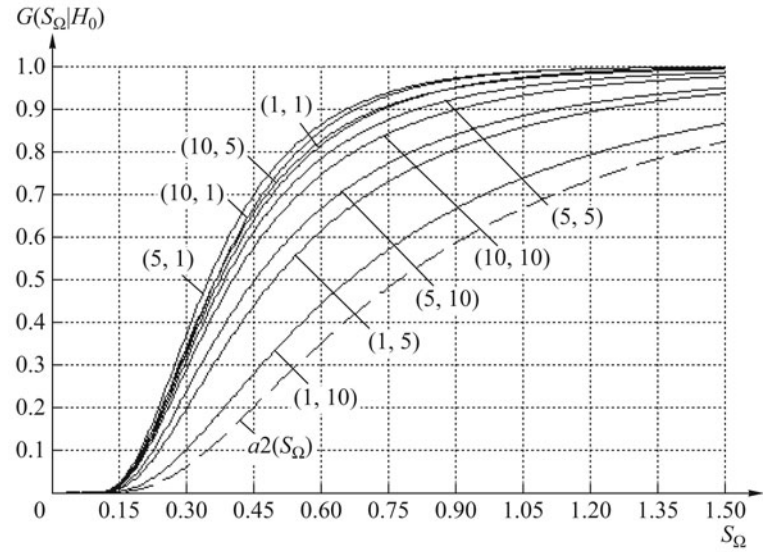
Studies of limit statistic distributions of nonparametric goodness-of-fit criteria for complex tested hypotheses were began in the work [36]. Later, various approaches have been tried on this problem: limit statistic distributions have been studied with analytic methods [37–49]; percentiles of these distributions have been constructed with statistical modeling [50–53]; formulas that give good approximations for small values of corresponding probabilities have been presented [54–56].

In the works [1–8], statistic distributions of nonparametric goodness-of-fit criteria were studied with statistical modeling. Further, based on the obtained empirical statistic distributions, percentile tables and approximate analytic models for statistic distribution laws were constructed.

3. STATISTIC DISTRIBUTION MODELS FOR GOODNESS-OF-FIT CRITERIA IN CASE OF THE INVERSE GAUSSIAN DISTRIBUTION

The density of the inverse Gaussian distribution looks like

$$f(x) = IG(\theta_0, \theta_1) = \left(\frac{\theta_1}{2\pi x^3}\right)^{1/2} \exp\left(-\frac{\theta_1 (x - \theta_0)^2}{2\theta_0^2 x}\right), \tag{4}$$



Anderson-Darling statistic distributions as functions of θ_0 and θ_1 for the case of estimating the maximal likelihood method parameters.

where parameters $\theta_0, \theta_1 \in (0, \infty)$ and $x \in (0, \infty)$. This law is especially interesting in reliability, longevity, and survival analysis problems, where there are reasons to believe that the fault intensity function is bell-shaped. Obviously, in this case the log-normal distribution is a natural alternative. We should also note that these two models can also compete with the three-parametric model of the generalized Weibull distribution (see, e.g., [57]) whose intensity function may assume all known forms, in particular, a bell-shaped one.

In testing complex hypotheses with respect to the inverse Gaussian law, distributions of the goodness-of-fit criteria statistic $G(S \mid H_0)$ depend on exact values of θ_0 and θ_1. This relation, for the Anderson-Darling criterion statistic distributions in the case of using MLE, is shown on the figure.

In this work, which continues and extends [58, 59], the problem of studying statistic distributions for testing complex hypotheses with respect to the inverse Gaussian law for nonparametric goodness-of-fit criteria statistics is extended to a large range of values for parameters θ_0 and θ_1 of the family (4). Percentile tables and statistic distribution models for Kolmogorov, Cramér–Mises–Smirnov ω^2, and Anderson–Darling Ω^2 criteria have been constructed by modeled samples for these statistics of size $N = 10^6$. For these values of N, the difference between the true statistic distribution law $G(S \mid H_0)$ and the modeled empirical $G_N(S \mid H_0)$ does not exceed 0.001 in absolute value. The values of criteria statistics have been computed by the samples of pseudorandom values generated according to the observed law $F(x, \theta)$ with sample size $n = 10^3$. In this situation, the distribution $G(S_n \mid H_0)$ virtually coincides with the limit distribution $G(S \mid H_0)$.

As a rule, the best models for the $G(S \mid H_0)$ distribution of Kolmogorov, Cramér–Mises–Smirnov ω^2, and Anderson–Darling Ω^2 statistics belong the family of beta distributions of the III type with density

$$B_{\mathrm{III}}(\theta_0, \theta_1, \theta_2, \theta_3, \theta_4) = \frac{\theta_2^{\theta_0}}{\theta_3 B(\theta_0, \theta_1)} \frac{\left(\dfrac{x-\theta_4}{\theta_3}\right)^{\theta_0-1} \left(1 - \dfrac{x-\theta_4}{\theta_3}\right)^{\theta_1-1}}{\left(1 + (\theta_2 - 1)\dfrac{x-\theta_4}{\theta_3}\right)^{\theta_0+\theta_1}}.$$

The found upper percentiles and constructed models for limit distributions of the Kolmogorov criterion statistic in case of computing MLE for both parameters of the law are presented in Table 1, for limit distributions of the Anderson–Darling criterion in Table 2, and for the Cramér–Mises–Smirnov criterion in Table 3.

Table 1. Percentiles and distributions for the Kolmogorov statistic for testing complex hypotheses with computing the MLEs of two parameters of an inverse Gaussian distribution

θ_0	θ_1	Percentiles			Statistic distribution model
		0.9	0.95	0.99	
1	1	0.910	0.998	1.180	$B_{\mathrm{III}}(5.9569; 6.7824; 3.0220; 1.6782; 0.2806)$
1	2	0.957	1.055	1.264	$B_{\mathrm{III}}(6.2372; 6.7506; 3.6809; 1.9867; 0.2800)$
1	3	0.997	1.105	1.342	$B_{\mathrm{III}}(6.6946; 6.3716; 4.5298; 2.2117; 0.2775)$
1	4	1.034	1.151	1.400	$B_{\mathrm{III}}(6.0585; 5.8419; 4.3611; 2.2007; 0.2883)$
1	5	1.066	1.189	1.450	$B_{\mathrm{III}}(6.1201; 5.4407; 4.5330; 2.2010; 0.2880)$
1	6	1.094	1.221	1.489	$B_{\mathrm{III}}(6.1759; 5.1904; 4.6124; 2.2013; 0.2878)$
1	7	1.121	1.250	1.525	$B_{\mathrm{III}}(6.2265; 4.9927; 4.6521; 2.2018; 0.2873)$
1	8	1.143	1.273	1.548	$B_{\mathrm{III}}(6.0718; 4.9852; 4.4776; 2.2284; 0.2896)$
1	9	1.159	1.292	1.565	$B_{\mathrm{III}}(5.6511; 4.9390; 4.0653; 2.1915; 0.2975)$
1	10	1.172	1.306	1.577	$B_{\mathrm{III}}(5.6247; 4.8832; 4.0106; 2.1894; 0.2995)$
2	1	0.882	0.964	1.137	$B_{\mathrm{III}}(6.3443; 7.5748; 2.9473; 1.6876; 0.2713)$
2	2	0.917	1.006	1.199	$B_{\mathrm{III}}(10.791; 6.9904; 5.6290; 1.9870; 0.2247)$
2	3	0.948	1.045	1.258	$B_{\mathrm{III}}(5.0737; 7.2993; 2.9576; 1.9859; 0.3000)$
2	4	0.977	1.082	1.319	$B_{\mathrm{III}}(5.4784; 7.0420; 3.5741; 2.2027; 0.2950)$
2	5	1.006	1.120	1.369	$B_{\mathrm{III}}(5.9645; 6.1578; 4.3007; 2.2159; 0.2900)$
2	6	1.032	1.152	1.408	$B_{\mathrm{III}}(7.6025; 5.4045; 5.7487; 2.2002; 0.2700)$
2	7	1.057	1.180	1.446	$B_{\mathrm{III}}(7.9396; 5.1500; 6.0298; 2.2004; 0.2650)$
2	8	1.079	1.206	1.476	$B_{\mathrm{III}}(5.9398; 5.1990; 4.6104; 2.2008; 0.2930)$
2	9	1.100	1.228	1.503	$B_{\mathrm{III}}(5.7955; 5.1024; 4.4620; 2.2011; 0.2950)$
2	10	1.120	1.250	1.528	$B_{\mathrm{III}}(5.9634; 4.9581; 4.5469; 2.1960; 0.2920)$
3	1	0.869	0.949	1.118	$B_{\mathrm{III}}(6.3357; 7.5977; 2.8564; 1.6279; 0.2706)$
3	2	0.897	0.984	1.168	$B_{\mathrm{III}}(6.5174; 7.4125; 3.2234; 1.7769; 0.2700)$
3	3	0.923	1.016	1.217	$B_{\mathrm{III}}(12.343; 6.5394; 6.7642; 1.9877; 0.2179)$
3	4	0.947	1.046	1.269	$B_{\mathrm{III}}(8.9671; 6.1291; 5.5576; 1.9890; 0.2500)$
3	5	0.971	1.078	1.318	$B_{\mathrm{III}}(10.2720; 6.0888; 6.8884; 2.1989; 0.2400)$
3	6	0.997	1.110	1.361	$B_{\mathrm{III}}(12.4552; 5.4731; 8.9051; 2.2027; 0.2300)$
3	7	1.019	1.137	1.395	$B_{\mathrm{III}}(14.7050; 5.1363; 10.7535; 2.2125; 0.2200)$
3	8	1.040	1.162	1.429	$B_{\mathrm{III}}(14.7958; 4.8912; 11.0081; 2.1998; 0.2200)$
3	9	1.061	1.187	1.457	$B_{\mathrm{III}}(15.9316; 4.7067; 11.8935; 2.2001; 0.2150)$
3	10	1.079	1.208	1.478	$B_{\mathrm{III}}(16.1109; 4.5707; 12.1176; 2.2003; 0.2150)$
4	1	0.862	0.941	1.107	$B_{\mathrm{III}}(6.6438; 7.7673; 2.9294; 1.6332; 0.2653)$
4	2	0.885	0.969	1.149	$B_{\mathrm{III}}(6.1888; 7.3858; 2.9758; 1.6875; 0.2750)$
4	3	0.907	0.997	1.189	$B_{\mathrm{III}}(6.9205; 7.8981; 3.6110; 1.9962; 0.2650)$
4	4	0.928	1.023	1.234	$B_{\mathrm{III}}(5.9780; 9.1512; 3.3926; 2.400; 0.2806)$
4	5	0.950	1.051	1.282	$B_{\mathrm{III}}(11.9075; 5.7447; 7.5201; 1.9905; 0.2300)$
4	6	0.971	1.079	1.323	$B_{\mathrm{III}}(11.9043; 5.8982; 8.1014; 2.2019; 0.2300)$
4	7	0.993	1.107	1.360	$B_{\mathrm{III}}(16.9484; 5.3410; 11.9176; 2.1990; 0.2100)$
4	8	1.013	1.131	1.391	$B_{\mathrm{III}}(17.0835; 5.1135; 12.3060; 2.2027; 0.2100)$
4	9	1.032	1.155	1.423	$B_{\mathrm{III}}(17.3084; 4.8961; 12.7239; 2.1996; 0.2100)$
4	10	1.051	1.177	1.449	$B_{\mathrm{III}}(17.2467; 4.7356; 12.8311; 2.1997; 0.2100)$
5	1	0.857	0.935	1.099	$B_{\mathrm{III}}(6.7006; 7.8312; 2.9541; 1.6335; 0.2651)$
5	2	0.877	0.960	1.135	$B_{\mathrm{III}}(9.00487; 6.9071; 4.2573; 1.6881; 0.2400)$
5	3	0.897	0.984	1.172	$B_{\mathrm{III}}(8.9213; 7.75024.5226; 2.0008; 0.2400)$
5	4	0.915	1.008	1.212	$B_{\mathrm{III}}(9.4154; 6.9359; 5.2226; 1.9946; 0.2400)$
5	5	0.934	1.033	1.2518	$B_{\mathrm{III}}(5.8431; 14.4730; 3.6811; 4.000; 0.2806)$
5	6	0.953	1.057	1.294	$B_{\mathrm{III}}(14.8173; 5.6823; 9.4427; 2.0492; 0.2150)$

Table 1. (Contd.)

θ_0	θ_1	Percentiles			Statistic distribution model
		0.9	0.95	0.99	
5	7	0.974	1.083	1.331	$B_{\mathrm{III}}(15.2826; 5.6389; 10.5512; 2.2057; 0.2150)$
5	8	0.993	1.109	1.364	$B_{\mathrm{III}}(20.7049; 5.2372; 14.5735; 2.200; 0.2000)$
5	9	1.011	1.130	1.392	$B_{\mathrm{III}}(20.6647; 5.0521; 14.7877; 2.2001; 0.2000)$
5	10	1.029	1.152	1.421	$B_{\mathrm{III}}(26.3791; 4.8072; 19.0811; 2.2027; 0.1900)$
6	1	0.854	0.932	1.093	$B_{\mathrm{III}}(6.6964; 7.8242; 2.9493; 1.6287; 0.2640)$
6	2	0.871	0.953	1.126	$B_{\mathrm{III}}(8.8962; 7.1069; 4.1210; 1.6892; 0.2400)$
6	3	0.888	0.974	1.159	$B_{\mathrm{III}}(9.1443; 6.9243; 4.5492; 1.7798; 0.2400)$
6	4	0.906	0.996	1.193	$B_{\mathrm{III}}(9.2454; 7.2428; 4.9786; 1.9991; 0.2400)$
6	5	0.923	1.018	1.228	$B_{\mathrm{III}}(12.5264; 6.3510; 7.1854; 1.9935; 0.2200)$
6	6	0.940	1.041	1.269	$B_{\mathrm{III}}(5.8883; 17.8508; 4.6295; 6.000; 0.2806)$
6	7	0.956	1.064	1.306	$B_{\mathrm{III}}(18.3816; 5.8741; 11.9529; 2.2081; 0.2000)$
6	8	0.977	1.088	1.343	$B_{\mathrm{III}}(23.3374; 5.4599; 15.6767; 2.2051; 0.1900)$
6	9	0.995	1.111	1.369	$B_{\mathrm{III}}(34.6437; 5.1591; 23.3905; 2.2019; 0.1750)$
6	10	1.011	1.131	1.395	$B_{\mathrm{III}}(39.1153; 4.9860; 26.4984; 2.1988; 0.1700)$
7	1	0.854	0.930	1.091	$B_{\mathrm{III}}(6.5586; 8.0638; 2.8375; 1.6313; 0.2673)$
7	2	0.867	0.948	1.119	$B_{\mathrm{III}}(7.1224; 7.6538; 3.2597; 1.6903; 0.2600)$
7	3	0.883	0.967	1.149	$B_{\mathrm{III}}(7.2034; 7.5737; 3.4800; 1.7800; 0.2600)$
7	4	0.898	0.987	1.180	$B_{\mathrm{III}}(7.3038; 7.9550; 3.8033; 2.0010; 0.2600)$
7	5	0.914	1.007	1.214	$B_{\mathrm{III}}(8.5307; 6.9961; 4.8484; 1.9974; 0.2500)$
7	6	0.930	1.028	1.250	$B_{\mathrm{III}}(15.1195; 5.9784; 8.9543; 1.9927; 0.2100)$
7	7	0.946	1.049	1.285	$B_{\mathrm{III}}(5.8391; 17.5429; 4.6421; 6.000; 0.2806)$
7	8	0.963	1.071	1.322	$B_{\mathrm{III}}(31.3194; 5.6956; 19.4931; 2.2085; 0.1700)$
7	9	0.980	1.094	1.353	$B_{\mathrm{III}}(42.2228; 5.0877; 30.0214; 2.2059; 0.1800)$
7	10	0.997	1.115	1.376	$B_{\mathrm{III}}(42.3418; 5.0662; 28.8200; 2.2033; 0.1700)$
8	1	0.854	0.930	1.091	$B_{\mathrm{III}}(6.7259; 7.9921; 2.8961; 1.6227; 0.2658)$
8	2	0.864	0.943	1.113	$B_{\mathrm{III}}(13.28101; 6.7409; 6.0703; 1.6914; 0.2100)$
8	3	0.878	0.961	1.140	$B_{\mathrm{III}}(13.1182; 6.5995; 6.1889; 1.7308; 0.2100)$
8	4	0.892	0.980	1.168	$B_{\mathrm{III}}(12.8930; 7.2771; 6.5124; 2.0016; 0.2100)$
8	5	0.907	0.998	1.201	$B_{\mathrm{III}}(14.5402; 6.6949; 7.78121; 2.0005; 0.2050)$
8	6	0.921	1.017	1.231	$B_{\mathrm{III}}(17.0918; 6.1135; 9.7622; 1.9962; 0.2000)$
8	7	0.936	1.037	1.267	$B_{\mathrm{III}}(21.3221; 5.5875; 13.0056; 1.9922; 0.1950)$
8	8	0.952	1.057	1.300	$B_{\mathrm{III}}(5.8759; 19.0528; 5.7987; 8.000; 0.2806)$
8	9	0.968	1.079	1.332	$B_{\mathrm{III}}(26.8192; 5.4765; 18.0105; 2.2088; 0.1850)$
8	10	0.984	1.100	1.360	$B_{\mathrm{III}}(33.2892; 5.1108; 23.7224; 2.2066; 0.1850)$
9	1	0.855	0.932	1.092	$B_{\mathrm{III}}(7.0181; 8.0734; 2.9699; 1.6366; 0.2620)$
9	2	0.861	0.940	1.108	$B_{\mathrm{III}}(14.8480; 6.7456; 6.6141; 1.6911; 0.2000)$
9	3	0.874	0.956	1.133	$B_{\mathrm{III}}(15.1905; 6.3828; 7.0328; 1.6895; 0.2000)$
9	4	0.887	0.973	1.161	$B_{\mathrm{III}}(14.5137; 7.3126; 7.1548; 2.0020; 0.2000)$
9	5	0.900	0.991	1.191	$B_{\mathrm{III}}(15.4545; 6.7896; 8.1115; 2.0011; 0.2000)$
9	6	0.914	1.008	1.220	$B_{\mathrm{III}}(16.8407; 6.2819; 9.4483; 1.9990; 0.2000)$
9	7	0.928	1.027	1.254	$B_{\mathrm{III}}(26.4103; 5.6444; 15.7438; 1.9953; 0.1850)$
9	8	0.943	1.046	1.285	$B_{\mathrm{III}}(27.0115; 5.3931; 16.5812; 1.9918; 0.1850)$
9	9	0.958	1.066	1.318	$B_{\mathrm{III}}(5.9761; 18.1201; 6.9936; 9.000; 0.2806)$
9	10	0.973	1.086	1.346	$B_{\mathrm{III}}(28.3999; 5.3460; 19.5082; 2.2090; 0.1850)$
10	1	0.858	0.935	1.094	$B_{\mathrm{III}}(6.3708; 8.3028; 2.6724; 1.6339; 0.2719)$
10	2	0.859	0.937	1.105	$B_{\mathrm{III}}(21.0180; 6.2702; 9.1530; 1.6325; 0.1800)$
10	3	0.871	0.952	1.128	$B_{\mathrm{III}}(20.8492; 6.2545; 9.3852; 1.6899; 0.1800)$
10	4	0.883	0.968	1.153	$B_{\mathrm{III}}(18.5209; 7.3068; 8.7206; 2.0024; 0.1800)$
10	5	0.896	0.985	1.179	$B_{\mathrm{III}}(20.1426; 6.8010; 10.0869; 2.0016; 0.1800)$
10	6	0.908	1.001	1.209	$B_{\mathrm{III}}(22.0824; 6.3600; 11.7113; 2.0008; 0.1800)$
10	7	0.921	1.018	1.239	$B_{\mathrm{III}}(25.7879; 5.8810; 14.6469; 1.9979; 0.1800)$
10	8	0.935	1.037	1.269	$B_{\mathrm{III}}(31.0404; 5.5468; 18.2144; 1.9946; 0.1750)$

Table 1. (Contd.)

θ_0	θ_1	Percentiles			Statistic distribution model
		0.9	0.95	0.99	
10	9	0.949	1.055	1.301	$B_{\rm III}(31.0159; 5.7740; 19.6302; 2.2075; 0.1750)$
10	10	0.964	1.075	1.331	$B_{\rm III}(5.9754; 17.6996; 7.5357; 9.5000; 0.2806)$

Table 2. Percentiles and distributions for the Anderson–Darling statistic for testing complex hypotheses with computing the MLEs of two parameters of an inverse Gaussian distribution

θ_0	θ_1	Percentiles			Statistic distribution model
		0.9	0.95	0.99	
1	1	0.737	0.895	1.313	$B_{\rm III}(6.1238; 3.1751; 14.0035; 2.400; 0.0698)$
1	2	0.843	1.059	1.735	$B_{\rm III}(6.0219; 3.6293; 30.2458; 6.200; 0.0675)$
1	3	0.961	1.255	2.196	$B_{\rm III}(6.5550; 3.0737; 46.299; 7.800; 0.0663)$
1	4	1.098	1.454	2.580	$B_{\rm III}(6.8209; 2.6705; 51.187; 7.700; 0.0667)$
1	5	1.228	1.647	2.869	$B_{\rm III}(6.6473; 2.4177; 51.272; 7.700; 0.0680)$
1	6	1.351	1.809	3.071	$B_{\rm III}(6.3414; 2.2618; 48.992; 7.700; 0.0716)$
1	7	1.467	1.950	3.257	$B_{\rm III}(5.9850; 2.2012; 44.595; 7.800; 0.0731)$
1	8	1.558	2.067	3.412	$B_{\rm III}(5.2130; 2.2645; 33.943; 7.700; 0.0735)$
1	9	1.634	2.156	3.490	$B_{\rm III}(5.3059; 2.2709; 32.7287; 7.800; 0.0714)$
1	10	1.698	2.224	3.572	$B_{\rm III}(5.9944; 2.0162; 30.5401; 6.000; 0.0708)$
2	1	0.690	0.829	1.181	$B_{\rm III}(6.0060; 3.5986; 12.5163; 2.400; 0.0675)$
2	2	0.754	0.925	1.428	$B_{\rm III}(5.5094; 4.2127; 21.2179; 5.200; 0.0698)$
2	3	0.826	1.041	1.764	$B_{\rm III}(6.4998; 3.7767; 45.2636; 8.6662; 0.0650)$
2	4	0.908	1.181	2.096	$B_{\rm III}(6.8554; 3.2620; 53.0151; 8.6676; 0.0650)$
2	5	1.007	1.337	2.424	$B_{\rm III}(7.4420; 2.8256; 66.5455; 9.0239; 0.0650)$
2	6	1.104	1.483	2.677	$B_{\rm III}(7.4598; 2.6158; 68.3448; 9.0242; 0.0650)$
2	7	1.203	1.624	2.864	$B_{\rm III}(7.4859; 2.4311; 69.9471; 9.0242; 0.0650)$
2	8	1.299	1.746	3.026	$B_{\rm III}(6.9268; 2.4170; 69.9882; 10.3004; 0.0650)$
2	9	1.387	1.854	3.153	$B_{\rm III}(6.9942; 2.3297; 69.5407; 10.2998; 0.0650)$
2	10	1.471	1.958	3.286	$B_{\rm III}(6.0278; 2.3810; 54.3565; 10.2991; 0.0650)$
3	1	0.673	0.805	1.134	$B_{\rm III}(5.8991; 3.7868; 11.8595; 2.400; 0.0672)$
3	2	0.718	0.873	1.311	$B_{\rm III}(6.1369; 4.8632; 35.6146; 8.6670; 0.0650)$
3	3	0.771	0.955	1.561	$B_{\rm III}(5.7293; 4.0413; 27.6069; 6.200; 0.0698)$
3	4	0.829	1.057	1.840	$B_{\rm III}(6.4965; 3.7146; 46.0939; 8.6656; 0.0650)$
3	5	0.900	1.176	2.132	$B_{\rm III}(7.4099; 3.1291; 61.1273; 8.6667; 0.0650)$
3	6	0.980	1.308	2.399	$B_{\rm III}(7.9874; 2.7921; 74.4933; 9.0231; 0.0650)$
3	7	1.063	1.427	2.600	$B_{\rm III}(8.0571; 2.7087; 72.5225; 9.0237; 0.0600)$
3	8	1.146	1.554	2.781	$B_{\rm III}(8.5422; 2.4144; 63.7751; 7.0000; 0.0600)$
3	9	1.227	1.667	2.935	$B_{\rm III}(8.7988; 2.2737; 68.9529; 7.2000; 0.0600)$
3	10	1.307	1.763	3.047	$B_{\rm III}(8.8004; 2.1754; 68.9524; 7.2000; 0.0600)$
4	1	0.664	0.793	1.110	$B_{\rm III}(5.8519; 3.8882; 11.5594; 2.400; 0.0671)$
4	2	0.698	0.845	1.238	$B_{\rm III}(5.3538; 5.3288; 22.9349; 7.0000; 0.0675)$
4	3	0.740	0.910	1.436	$B_{\rm III}(5.6387; 4.5708; 27.5081; 7.0000; 0.0675)$
4	4	0.787	0.986	1.686	$B_{\rm III}(5.7589; 4.09340; 39.9826; 9.000; 0.0698)$
4	5	0.842	1.084	1.934	$B_{\rm III}(6.5537; 3.4655; 46.4219; 8.0000; 0.0675)$
4	6	0.904	1.190	2.194	$B_{\rm III}(6.8253; 3.1718; 61.2636; 9.5000; 0.0675)$
4	7	0.974	1.307	2.419	$B_{\rm III}(7.4076; 2.8129; 69.3032; 9.0224; 0.0675)$
4	8	1.048	1.412	2.592	$B_{\rm III}(7.4062; 2.6410; 71.0431; 9.0231; 0.0675)$
4	9	1.123	1.524	2.760	$B_{\rm III}(7.4858; 2.4792; 73.6341; 9.0235; 0.0675)$
4	10	1.195	1.628	2.897	$B_{\rm III}(7.4411; 2.3667; 73.6365; 9.0239; 0.0675)$
5	1	0.659	0.786	1.096	$B_{\rm III}(5.8800; 3.9339; 11.5621; 2.400; 0.0671)$
5	2	0.687	0.827	0.827	$B_{\rm III}(5.3832; 5.5916; 27.7521; 8.6678; 0.0675)$
5	3	0.720	0.881	1.360	$B_{\rm III}(5.5318; 5.0043; 31.1395; 8.6673; 0.0675)$
5	4	0.760	0.944	1.571	$B_{\rm III}(5.8814; 4.3301; 37.6029; 8.6663; 0.0675)$

Table 2. (Contd.)

θ_0	θ_1	Percentiles			Statistic distribution model
		0.9	0.95	0.99	
5	5	0.803	1.023	1.799	$B_{\mathrm{III}}(6.1596; 3.8709; 60.9340; 12.000; 0.0698)$
5	6	0.856	1.113	2.035	$B_{\mathrm{III}}(6.6082; 3.3950; 51.7023; 8.6650; 0.0675)$
5	7	0.917	1.220	2.268	$B_{\mathrm{III}}(7.2066; 2.9940; 65.5245; 9.0213; 0.0680)$
5	8	0.979	1.318	1.318	$B_{\mathrm{III}}(7.3289; 2.7883; 69.4323; 9.0220; 0.0680)$
5	9	1.044	1.414	2.602	$B_{\mathrm{III}}(7.4884; 2.6173; 73.2183; 9.0226; 0.0680)$
5	10	1.114	1.514	2.757	$B_{\mathrm{III}}(7.4934; 2.4672; 75.0000; 9.0231; 0.0680)$
6	1	0.655	0.781	1.085	$B_{\mathrm{III}}(5.7986; 3.9980; 11.2400; 2.400; 0.0671)$
6	2	0.679	0.815	1.174	$B_{\mathrm{III}}(5.2935; 5.8182; 26.4298; 8.6680; 0.0680)$
6	3	0.707	0.860	1.305	$B_{\mathrm{III}}(5.3441; 5.3154; 28.5532; 8.6676; 0.0680)$
6	4	0.740	0.915	1.489	$B_{\mathrm{III}}(5.7947; 4.5262; 35.9854; 8.6670; 0.0680)$
6	5	0.778	0.979	1.694	$B_{\mathrm{III}}(6.1074; 4.0045; 42.1634; 8.6662; 0.0680)$
6	6	0.822	1.0587	1.907	$B_{\mathrm{III}}(6.1637; 3.6120; 46.396; 8.6652; 0.0698)$
6	7	0.872	1.145	2.123	$B_{\mathrm{III}}(6.7575; 3.1650; 57.3316; 8.6649; 0.0700)$
6	8	0.929	1.246	2.340	$B_{\mathrm{III}}(7.1614; 2.8755; 68.2312; 9.0211; 0.0700)$
6	9	0.988	1.333	2.474	$B_{\mathrm{III}}(7.2146; 2.7131; 70.7516; 9.0217; 0.0700)$
6	10	1.049	1.422	2.628	$B_{\mathrm{III}}(7.2844; 2.5752; 72.9798; 9.0222; 0.0700)$
7	1	0.655	0.780	1.081	$B_{\mathrm{III}}(5.8421; 4.0362; 11.1704; 2.400; 0.0672)$
7	2	0.673	0.807	1.156	$B_{\mathrm{III}}(5.4561; 5.9720; 26.4093; 8.6680; 0.0650)$
7	3	0.697	0.846	1.270	$B_{\mathrm{III}}(5.5685; 5.4358; 29.1024; 8.6678; 0.0650)$
7	4	0.726	0.894	1.427	$B_{\mathrm{III}}(5.9802; 4.7226; 35.6184; 8.6673; 0.0650)$
7	5	0.759	0.948	1.617	$B_{\mathrm{III}}(6.3225; 4.1978; 41.7939; 8.6668; 0.0650)$
7	6	0.779	0.984	1.734	$B_{\mathrm{III}}(6.8608; 3.8616; 49.1300; 8.6666; 0.0640)$
7	7	0.841	1.096	2.018	$B_{\mathrm{III}}(6.1138; 3.6208; 63.498; 12.000; 0.0698)$
7	8	0.891	1.181	2.217	$B_{\mathrm{III}}(7.1968; 3.1822; 61.7629; 9.0202; 0.0650)$
7	9	0.943	1.272	2.388	$B_{\mathrm{III}}(7.5770; 2.9005; 69.9780; 9.0208; 0.0650)$
7	10	1.001	1.357	2.515	$B_{\mathrm{III}}(7.7412; 2.7373; 73.7918; 9.0214; 0.0650)$
8	1	0.660	0.785	1.081	$B_{\mathrm{III}}(6.1227; 4.0360; 11.5336; 2.400; 0.0666)$
8	2	0.668	0.801	1.138	$B_{\mathrm{III}}(5.5121; 5.8006; 27.9640; 8.6681; 0.0670)$
8	3	0.690	0.836	1.241	$B_{\mathrm{III}}(5.4002; 5.5554; 27.9506; 8.6679; 0.0670)$
8	4	0.715	0.877	1.380	$B_{\mathrm{III}}(5.6470; 4.9453; 32.4456; 8.6676; 0.0670)$
8	5	0.745	0.926	1.550	$B_{\mathrm{III}}(5.9674; 4.4102; 37.9915; 8.6671; 0.0670)$
8	6	0.779	0.984	1.734	$B_{\mathrm{III}}(6.4110; 3.8911; 45.8150; 8.6666; 0.0670)$
8	7	0.817	1.054	1.929	$B_{\mathrm{III}}(7.0348; 3.4829; 55.3483; 8.6659; 0.0660)$
8	8	0.859	1.129	2.106	$B_{\mathrm{III}}(6.4429; 3.4105; 71.057; 12.000; 0.0698)$
8	9	0.910	1.220	2.295	$B_{\mathrm{III}}(7.8320; 2.8953; 75.0000; 9.0201; 0.0670)$
8	10	0.961	1.300	2.435	$B_{\mathrm{III}}(7.6032; 2.8065; 72.8901; 9.0207; 0.0670)$
9	1	0.667	0.794	1.091	$B_{\mathrm{III}}(6.0785; 4.0967; 11.0332; 2.400; 0.0672)$
9	2	0.664	0.796	1.126	$B_{\mathrm{III}}(5.2061; 6.2168; 24.4769; 8.6680; 0.0670)$
9	3	0.684	0.827	1.210	$B_{\mathrm{III}}(5.3219; 5.7271; 26.7940; 8.6680; 0.0670)$
9	4	0.707	0.865	1.345	$B_{\mathrm{III}}(5.4618; 5.1961; 29.9012; 8.6677; 0.0670)$
9	5	0.734	0.910	1.496	$B_{\mathrm{III}}(5.9892; 4.487; 37.8576; 8.6674; 0.0670)$
9	6	0.763	0.959	1.666	$B_{\mathrm{III}}(6.2913; 4.0565; 43.4772; 8.6670; 0.0670)$
9	7	0.797	1.023	1.855	$B_{\mathrm{III}}(6.4827; 3.7214; 48.0168; 8.6664; 0.0670)$
9	8	0.836	1.092	2.036	$B_{\mathrm{III}}(6.7595; 3.4190; 53.6318; 8.6659; 0.0670)$
9	9	0.881	1.170	2.211	$B_{\mathrm{III}}(6.3746; 3.3201; 74.534; 12.500; 0.0698)$
9	10	0.927	1.252	2.362	$B_{\mathrm{III}}(7.7636; 2.8465; 75.0000; 9.0201; 0.0670)$
10	1	0.678	0.808	1.108	$B_{\mathrm{III}}(5.8108; 4.1511; 10.1199; 2.400; 0.0676)$
10	2	0.662	0.792	1.119	$B_{\mathrm{III}}(5.4827; 5.5338; 20.0888; 6.0043; 0.0671)$
10	3	0.679	0.820	1.192	$B_{\mathrm{III}}(5.2739; 5.8610; 26.0201; 8.6680; 0.0670)$
10	4	0.700	0.854	1.312	$B_{\mathrm{III}}(5.4167; 5.3561; 28.8890; 8.6678; 0.0670)$
10	5	0.724	0.895	1.455	$B_{\mathrm{III}}(5.8460; 4.6549; 35.7438; 8.6676; 0.0670)$
10	6	0.751	0.939	1.616	$B_{\mathrm{III}}(6.1462; 4.2118; 41.1210; 8.6672; 0.0670)$

Table 2. (Contd.)

θ_0	θ_1	Percentiles			Statistic distribution model
		0.9	0.95	0.99	
10	7	0.782	0.997	1.785	$B_{\mathrm{III}}(6.5669; 3.7838; 48.4614; 8.6668; 0.0670)$
10	8	0.818	1.059	1.959	$B_{\mathrm{III}}(7.0172; 3.4172; 56.7255; 8.6663; 0.0670)$
10	9	0.855	1.128	2.120	$B_{\mathrm{III}}(6.4784; 3.4405; 74.0853; 12.5000; 0.0692)$
10	10	0.902	1.210	2.295	$B_{\mathrm{III}}(6.2788; 3.2908; 73.169; 12.500; 0.0698)$

Table 3. Percentiles and distributions for the Cramér–Mises–Smirnov statistic for testing complex hypotheses with computing the MLEs of two parameters of an inverse Gaussian distribution

θ_0	θ_1	Percentiles			Statistic distribution model
		0.9	0.95	0.99	
1	1	0.130	0.162	0.246	$B_{\mathrm{III}}(4.7725; 2.8933; 15.4899; 0.500; 0.0086)$
1	2	0.150	0.194	0.321	$B_{\mathrm{III}}(4.7817; 2.7916; 24.1428; 0.800; 0.0086)$
1	3	0.173	0.229	0.408	$B_{\mathrm{III}}(5.0019; 2.4825; 33.657; 1.000; 0.0086)$
1	4	0.195	0.265	0.481	$B_{\mathrm{III}}(4.8570; 2.3136; 39.4252; 1.200; 0.0087)$
1	5	0.218	0.298	0.537	$B_{\mathrm{III}}(4.8192; 2.1138; 40.1445; 1.200; 0.0088)$
1	6	0.239	0.328	0.575	$B_{\mathrm{III}}(4.7926; 1.9813; 40.1460; 1.200; 0.0089)$
1	7	0.260	0.355	0.613	$B_{\mathrm{III}}(4.3522; 1.9184; 38.5472; 1.300; 0.0097)$
1	8	0.277	0.378	0.645	$B_{\mathrm{III}}(4.1660; 1.9401; 36.7953; 1.400; 0.0097)$
1	9	0.290	0.395	0.663	$B_{\mathrm{III}}(1.7177; 2.5911; 14.9047; 2.0049; 0.020)$
1	10	0.302	0.409	0.682	$B_{\mathrm{III}}(1.7373; 2.6035; 14.6773; 2.0578; 0.020)$
2	1	0.119	0.146	0.216	$B_{\mathrm{III}}(5.5933; 2.8761; 15.0784; 0.400; 0.0075)$
2	2	0.132	0.167	0.265	$B_{\mathrm{III}}(4.7336; 2.9815; 17.9631; 0.600; 0.0086)$
2	3	0.146	0.189	0.323	$B_{\mathrm{III}}(5.0307; 2.7954; 26.1410; 0.800; 0.0084)$
2	4	0.162	0.214	0.387	$B_{\mathrm{III}}(5.2560; 2.5116; 32.9308; 0.900; 0.0084)$
2	5	0.178	0.242	0.446	$B_{\mathrm{III}}(5.4485; 2.2935; 39.9250; 1.000; 0.0084)$
2	6	0.195	0.267	0.498	$B_{\mathrm{III}}(5.3297; 2.1938; 42.6895; 1.100; 0.0084)$
2	7	0.212	0.292	0.534	$B_{\mathrm{III}}(5.3797; 2.0580; 45.8508; 1.150; 0.0084)$
2	8	0.228	0.316	0.565	$B_{\mathrm{III}}(5.4058; 1.9526; 46.3260; 1.150; 0.0084)$
2	9	0.244	0.336	0.590	$B_{\mathrm{III}}(5.3765; 1.8676; 45.9372; 1.150; 0.0084)$
2	10	0.259	0.356	0.618	$B_{\mathrm{III}}(5.3119; 1.8093; 45.0334; 1.160; 0.0084)$
3	1	0.114	0.140	0.204	$B_{\mathrm{III}}(4.6237; 3.1152; 11.9551; 0.400; 0.0087)$
3	2	0.124	0.155	0.239	$B_{\mathrm{III}}(4.5231; 3.3129; 15.9686; 0.600; 0.0087)$
3	3	0.135	0.172	0.284	$B_{\mathrm{III}}(4.7701; 3.0644; 22.1622; 0.750; 0.0087)$
3	4	0.146	0.191	0.336	$B_{\mathrm{III}}(4.9007; 2.7581; 26.0854; 0.8000; 0.0087)$
3	5	0.159	0.212	0.391	$B_{\mathrm{III}}(5.1313; 2.4877; 31.2188; 0.8500; 0.0087)$
3	6	0.174	0.235	0.440	$B_{\mathrm{III}}(5.1565; 2.4026; 40.7673; 1.1000; 0.0087)$
3	7	0.187	0.256	0.483	$B_{\mathrm{III}}(5.2195; 2.2343; 42.6422; 1.1000; 0.0087)$
3	8	0.202	0.278	0.520	$B_{\mathrm{III}}(5.2685; 2.1243; 47.574; 1.2000; 0.0087)$
3	9	0.216	0.299	0.548	$B_{\mathrm{III}}(5.2645; 2.0191; 47.9732; 1.2000; 0.0087)$
3	10	0.229	0.318	0.570	$B_{\mathrm{III}}(5.2768; 1.9423; 48.0147; 1.2000; 0.0087)$
4	1	0.112	0.137	0.198	$B_{\mathrm{III}}(4.5782; 3.2113; 11.6509; 0.4000; 0.0087)$
4	2	0.120	0.148	0.224	$B_{\mathrm{III}}(4.6470; 3.0937; 13.7864; 0.4645; 0.0087)$
4	3	0.128	0.162	0.262	$B_{\mathrm{III}}(4.7447; 3.0421; 18.1139; 0.6000; 0.0087)$
4	4	0.137	0.177	0.304	$B_{\mathrm{III}}(4.6410; 3.2743; 29.7398; 1.100; 0.0087)$
4	5	0.148	0.194	0.354	$B_{\mathrm{III}}(4.9182; 2.9156; 37.7830; 1.2000; 0.0087)$
4	6	0.159	0.213	0.402	$B_{\mathrm{III}}(4.9451; 2.8167; 54.9745; 1.7197; 0.0087)$
4	7	0.172	0.234	0.442	$B_{\mathrm{III}}(4.9463; 2.6275; 56.8175; 1.7197; 0.0087)$
4	8	0.184	0.253	0.479	$B_{\mathrm{III}}(5.2218; 2.4515; 61.8564; 1.7198; 0.0083)$
4	9	0.197	0.272	0.512	$B_{\mathrm{III}}(5.1854; 2.3115; 63.0610; 1.7198; 0.0085)$
4	10	0.210	0.291	0.542	$B_{\mathrm{III}}(5.1470; 2.2041; 63.2902; 1.7198; 0.0085)$
5	1	0.111	0.135	0.194	$B_{\mathrm{III}}(4.5239; 3.2857; 11.3509; 0.4000; 0.0087)$
5	2	0.117	0.144	0.216	$B_{\mathrm{III}}(4.3756; 3.6726; 14.4290; 0.6000; 0.0087)$

Table 3. (Contd.)

θ_0	θ_1	Percentiles			Statistic distribution model
		0.9	0.95	0.99	
5	3	0.124	0.156	0.247	$B_{\mathrm{III}}(4.4097; 3.6237; 19.2576; 0.8000; 0.0087)$
5	4	0.132	0.168	0.283	$B_{\mathrm{III}}(4.5663; 3.4838; 28.1077; 1.1000; 0.0087)$
5	5	0.140	0.183	0.325	$B_{\mathrm{III}}(4.6087; 3.2941; 37.3032; 1.4000; 0.0087)$
5	6	0.150	0.199	0.369	$B_{\mathrm{III}}(4.6721; 3.0859; 44.9449; 1.6000; 0.0087)$
5	7	0.161	0.217	0.415	$B_{\mathrm{III}}(4.9820; 2.8196; 64.4405; 2.0000; 0.0086)$
5	8	0.172	0.235	0.448	$B_{\mathrm{III}}(4.9695; 2.6609; 69.3170; 2.1000; 0.0086)$
5	9	0.183	0.252	0.483	$B_{\mathrm{III}}(4.9353; 2.5489; 72.7848; 2.2000; 0.0086)$
5	10	0.195	0.270	0.511	$B_{\mathrm{III}}(4.9489; 2.4090; 74.7198; 2.2000; 0.0086)$
6	1	0.110	0.133	0.192	$B_{\mathrm{III}}(4.5010; 3.2285; 10.9781; 0.380; 0.0088)$
6	2	0.115	0.141	0.210	$B_{\mathrm{III}}(4.5811; 3.3997; 13.7378; 0.5000; 0.0087)$
6	3	0.121	0.151	0.235	$B_{\mathrm{III}}(4.5344; 3.3779; 16.0429; 0.6000; 0.0087)$
6	4	0.128	0.162	0.269	$B_{\mathrm{III}}(4.7010; 3.1896; 20.2456; 0.7000; 0.0087)$
6	5	0.135	0.174	0.305	$B_{\mathrm{III}}(4.7778; 3.1179; 26.6577; 0.9000; 0.0087)$
6	6	0.143	0.188	0.348	$B_{\mathrm{III}}(4.6979; 3.2496; 45.496; 1.6546; 0.0087)$
6	7	0.152	0.205	0.389	$B_{\mathrm{III}}(4.7377; 3.0239; 52.3921; 1.8000; 0.0087)$
6	8	0.163	0.221	0.426	$B_{\mathrm{III}}(4.8088; 2.8386; 61.3983; 2.0000; 0.0087)$
6	9	0.173	0.237	0.453	$B_{\mathrm{III}}(4.8848; 2.6529; 70.0431; 2.1500; 0.0087)$
6	10	0.183	0.253	0.487	$B_{\mathrm{III}}(4.9708; 2.5260; 74.5301; 2.2000; 0.0087)$
7	1	0.110	0.133	0.191	$B_{\mathrm{III}}(4.3709; 3.2673; 10.5551; 0.3800; 0.0090)$
7	2	0.114	0.140	0.207	$B_{\mathrm{III}}(4.5828; 3.2976; 12.7789; 0.4500; 0.0087)$
7	3	0.119	0.148	0.227	$B_{\mathrm{III}}(4.4677; 3.4969; 15.4121; 0.6000; 0.0087)$
7	4	0.125	0.158	0.258	$B_{\mathrm{III}}(4.6214; 3.3308; 19.2746; 0.7000; 0.0087)$
7	5	0.131	0.168	0.289	$B_{\mathrm{III}}(4.7195; 3.2125; 24.4491; 0.8500; 0.0087)$
7	6	0.138	0.181	0.327	$B_{\mathrm{III}}(4.6567; 3.3731; 44.3195; 1.6546; 0.0087)$
7	7	0.146	0.194	0.364	$B_{\mathrm{III}}(4.8434; 3.0794; 51.559; 1.7197; 0.0087)$
7	8	0.155	0.209	0.405	$B_{\mathrm{III}}(4.7910; 2.9523; 57.1575; 1.9000; 0.0087)$
7	9	0.165	0.225	0.436	$B_{\mathrm{III}}(4.8713; 2.7903; 72.7647; 2.3000; 0.0086)$
7	10	0.175	0.241	0.463	$B_{\mathrm{III}}(4.2642; 2.6670; 74.6989; 2.6000; 0.01)$
8	1	0.110	0.134	0.191	$B_{\mathrm{III}}(4.4757; 3.2680; 10.7031; 0.380; 0.0090)$
8	2	0.113	0.138	0.203	$B_{\mathrm{III}}(4.5960; 3.1619; 11.8642; 0.4000; 0.0087)$
8	3	0.117	0.145	0.221	$B_{\mathrm{III}}(4.4434; 3.5777; 15.1135; 0.6000; 0.0087)$
8	4	0.122	0.154	0.247	$B_{\mathrm{III}}(4.4746; 3.6125; 19.9058; 0.8000; 0.0087)$
8	5	0.128	0.164	0.278	$B_{\mathrm{III}}(4.5738; 3.4174; 23.8064; 0.9000; 0.0087)$
8	6	0.135	0.174	0.312	$B_{\mathrm{III}}(4.7719; 3.2182; 31.9744; 1.1000; 0.0087)$
8	7	0.142	0.187	0.349	$B_{\mathrm{III}}(4.7862; 3.1033; 40.0738; 1.3500; 0.0087)$
8	8	0.149	0.200	0.384	$B_{\mathrm{III}}(4.9118; 2.9697; 53.987; 1.7197; 0.0087)$
8	9	0.159	0.215	0.420	$B_{\mathrm{III}}(4.9553; 2.8177; 68.7411; 2.1000; 0.0087)$
8	10	0.167	0.230	0.447	$B_{\mathrm{III}}(4.8712; 2.7161; 74.7643; 2.3000; 0.0087)$
9	1	0.111	0.135	0.192	$B_{\mathrm{III}}(4.3128; 3.3456; 10.2390; 0.390; 0.0092)$
9	2	0.112	0.137	0.200	$B_{\mathrm{III}}(4.6271; 3.1831; 11.9529; 0.4000; 0.0087)$
9	3	0.116	0.143	0.217	$B_{\mathrm{III}}(4.3927; 3.6661; 14.6644; 0.6000; 0.0087)$
9	4	0.121	0.151	0.240	$B_{\mathrm{III}}(4.3715; 3.7463; 18.8452; 0.8000; 0.0087)$
9	5	0.126	0.160	0.269	$B_{\mathrm{III}}(4.5116; 3.6109; 25.0174; 1.0000; 0.0087)$
9	6	0.131	0.169	0.169	$B_{\mathrm{III}}(4.6314; 3.4428; 31.9594; 1.2000; 0.0087)$
9	7	0.138	0.181	0.333	$B_{\mathrm{III}}(4.7526; 3.2398; 40.0792; 1.4000; 0.0087)$
9	8	0.145	0.193	0.368	$B_{\mathrm{III}}(4.9123; 3.0606; 49.4625; 1.6000; 0.0087)$
9	9	0.153	0.207	0.403	$B_{\mathrm{III}}(5.1040; 2.8347; 58.579; 1.7197; 0.0086)$
9	10	0.161	0.221	0.432	$B_{\mathrm{III}}(4.9243; 2.7838; 65.2796; 2.0000; 0.0087)$
10	1	0.112	0.137	0.195	$B_{\mathrm{III}}(4.5358; 3.2996; 10.6858; 0.390; 0.0088)$
10	2	0.111	0.136	0.198	$B_{\mathrm{III}}(4.6058; 3.2130; 11.8374; 0.4000; 0.0087)$
10	3	0.115	0.142	0.213	$B_{\mathrm{III}}(4.4094; 3.7077; 14.6643; 0.6000; 0.0087)$
10	4	0.119	0.149	0.233	$B_{\mathrm{III}}(4.2916; 3.9108; 17.7709; 0.8000; 0.0087)$

Table 3. (Contd.)

θ_0	θ_1	Percentiles			Statistic distribution model
		0.9	0.95	0.99	
10	5	0.124	0.157	0.261	$B_{\text{III}}(4.4707; 3.7166; 24.2703; 1.0000; 0.0087)$
10	6	0.129	0.165	0.288	$B_{\text{III}}(4.5864; 3.5556; 30.9135; 1.2000; 0.0087)$
10	7	0.135	0.176	0.320	$B_{\text{III}}(4.6398; 3.3941; 37.6304; 1.4000; 0.0087)$
10	8	0.141	0.187	0.354	$B_{\text{III}}(4.8071; 3.1823; 48.6216; 1.6546; 0.0087)$
10	9	0.148	0.199	0.388	$B_{\text{III}}(4.8468; 3.0568; 60.6841; 2.0000; 0.0087)$
10	10	0.156	0.213	0.420	$B_{\text{III}}(5.0443; 2.7436; 59.626; 1.7197; 0.0088)$

4. CONCLUSION

The inverse Gaussian distribution finds applications in reliability, longevity, and survival analysis problems. For certain combinations of parameters, the distribution functions of the inverse Gaussian distribution are close enough to log-normal distribution functions. In this case, these two laws are hard to distinguish with goodness-of-fit criteria, especially for bounded sample sizes. At the same time, densities of these laws are notably different, and the corresponding intensity (fault) functions are bell-shaped in both cases and very much different.

Table 4 show the obtained power estimates for the criteria we have considered in this work and the Pearson χ^2 criterion for testing a complex hypothesis that the sample belongs to an inverse Gaussian distribution with parameters $\theta_0 = 2$, $\theta_1 = 2$ (estimating both parameters) against a

Table 4. Powers of the goodness-of-fit criteria for testing a complex hypothesis H_0 (inverse Gaussian distribution) against a hypothesis H_1 (log-normal) for a given probability of the error of the first type α

α	$n = 100$	$n = 200$	$n = 300$	$n = 400$	$n = 500$	$n = 600$
Power of the Anderson–Darling Ω^2 criterion						
0.1	0.246	0.363	0.473	0.571	0.656	0.897
0.05	0.155	0.252	0.353	0.446	0.534	0.829
0.01	0.041	0.087	0.136	0.202	0.264	0.590
Power of the Cramér–Mises–Smirnov ω^2 criterion						
0.1	0.236	0.346	0.450	0.544	0.622	0.861
0.05	0.149	0.241	0.329	0.417	0.493	0.776
0.01	0.040	0.084	0.129	0.186	0.247	0.524
Power of the Kolmogorov criterion						
0.1	0.211	0.299	0.388	0.466	0.542	0.787
0.05	0.129	0.202	0.276	0.346	0.612	0.678
0.01	0.034	0.069	0.102	0.145	0.187	0.420
Power of the Pearson χ^2 criterion for Neyman–Pearson intervals ($k = 5$)						
0.1	0.234	0.322	0.429	0.517	0.598	0.848
0.05	0.137	0.200	0.297	0.379	0.449	0.761
0.01	0.030	0.056	0.080	0.134	0.188	0.499
Power of the Pearson χ^2 criterion for $k = 5$ and uniform grouping						
0.1	0.149	0.195	0.255	0.301	0.344	0.584
0.05	0.080	0.117	0.154	0.193	0.227	0.448
0.01	0.020	0.035	0.046	0.059	0.079	0.210
Power of the Pearson χ^2 criterion for $k = 10$ and uniform grouping						
0.1	0.134	0.176	0.212	0.249	0.303	0.499
0.05	0.078	0.100	0.126	0.151	0.187	0.372
0.01	0.019	0.027	0.039	0.041	0.061	0.165

competing hypothesis that the sample belongs to the log-normal law with density

$$f(x) = \frac{1}{x\theta_0\sqrt{2\pi}} e^{-(lnx-\theta_1)^2/(2\theta_0^2)}$$

and parameters $\theta_0 = 0.82538$, $\theta_1 = 0.33102$.

As we expected [15–17], among non-parametric criteria the Anderson–Darling criterion turned out to be the most powerful.

Estimates of the power of Pearson χ^2 are shown for uniform grouping with the number of intervals $k = 10$ and $k = 5$, and also with the so-called Neyman–Pearson intervals [60], when interval bounds for computing the statistic are taken to be the intersection points for the competing densities (in this case, four points and five intervals). We should note that for testing complex hypotheses, it is better to use modified criteria of the χ^2 type, e.g., the Rao–Robson–Nikulin criterion [61–63], since in this situation they are more powerful than Pearson's χ^2 [15, 17].

Thus, using computerized methods of studying statistical patterns based on statistical modeling of empirical statistic distributions and subsequent analysis of these distributions, we have constructed statistic distribution models for nonparametric goodness-of-fit criteria: Kolmogorov, Cramér–Mises–Smirnov ω^2, and Anderson–Darling Ω^2, for testing complex hypothesis with respect to the inverse Gaussian law. Our results allow for a correct application of nonparametric goodness-of-fit criteria for testing the corresponding complex hypotheses.

Based on the results partially presented in this work and the developed software, a course "Computer technologies of data mining and studying statistical patterns" has been developed [64,65]; it is deployed at the Department of Applied Mathematics of the Novosibirsk State Technical University.

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