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ANALYSIS AND SYNTHESIS OF SIGNALS AND IMAGES

Solving Problems of Using Some Nonparametric Goodness-of-Fit Tests

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Abstract— Estimates are given of the power of the Kuiper and Watson goodness-of-fit tests and three Zhang tests with the Z_A, Z_C, and Z_K statistics with respect to some pairs of competing laws in testing simple and composite hypotheses. The powers of these tests are compared with the powers of the Kolmogorov, Cramer–von Mises–Smirnov, and Anderson–Darling tests. Statistic distribution models and tables of percentage points are constructed which allow the Kuiper and Watson goodness-of-fit tests to be used to test composite hypotheses about the goodness of fit of samples against various parametric distribution laws. An interactive simulation method is proposed that allows constructing and using distributions of test statistics in solving problems of statistical analysis.

Keywords: nonparametric goodness-of-fit tests, Kuiper test, Watson test, Zhang tests, test power, simple and composite hypotheses.

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INTRODUCTION

The list of goodness-of-fit tests used in the statistical analysis of experimental results is limited, as a rule, to Pearson's χ^2 test and two or three nonparametric tests: usually, the Kolmogorov, Cramer–von Mises–Smirnov, and Anderson–Darling tests. Their applications are often accompanied by errors that lead to incorrect inferences.

Using goodness-of-fit tests, one needs to distinguish between simple and composite hypotheses. A simple hypothesis is of the form H_0: $F(x) = F(x, \theta)$, where $F(x, \theta)$ is a known theoretical probability distribution function with a known scalar or vector parameter θ. In testing simple hypotheses, the nonparametric goodness-of-fit tests are distribution free, i.e., if the hypothesis to be tested is valid, the distributions $G(S \,|\, H_0)$ of their statistics are independent of the law $F(x, \theta)$ against which the goodness of fit is tested.

In the case of testing composite hypotheses of the form H_0: $F(x) \in \{F(x, \theta),\ \theta \in \Theta\}$, where the estimate $\hat{\theta}$ of the scalar or vector distribution parameter $F(x, \theta)$ is calculated from the same sample, nonparametric goodness-of-fit tests lose the distribution-free property. In testing of composite hypotheses, the conditional distributions of the statistics $G(S \,|\, H_0)$ depend on several factors: type of the observed law $F(x, \theta)$ corresponding to the valid tested hypothesis H_0; estimated parameter type and number of estimated parameters; in some cases, on a particular value of the parameter (for example, in the case of families of gamma and beta distributions); parameter estimation method. The differences in the distributions of the same statistic in testing simple and composite hypotheses are so significant that this fact cannot be neglected in any case.

In this study we had two objectives: (1) to draw attention to the Kuiper [1], Watson [2, 3], and Zhang [4–7] tests, which, by virtue of objective reasons and the lack of evidence-based recommendations are unfortunately little used in applied data analysis; (2) to solve the existing problems in the use of these and other tests for testing composite hypotheses, and some of them for testing simple hypotheses.

We emphasize that the previously reported results on the classical Kuiper and Watson tests and the test discussed in [4–7] are related only to testing simple hypotheses.

Table 1. Upper percentage points of the distributions of the Kuiper and Watson test statistics and their corresponding probabilities of the form $P(S > S_\alpha \mid H_0)$ calculated in accordance with the theoretical laws and from the results of statistical modeling

Percentage points and distributions of statistics	α				
	0.15	0.10	0.05	0.025	0.01
Percentage points of Kuiper statistics [10]	1.537	1.620	1.747	1.862	2.001
Distribution (2)	0.149945	0.099797	0.050075	0.025067	0.009994
Simulation result	0.149850	0.099636	0.050060	0.025006	0.009942
Distribution (5)	0.150283	0.100049	0.050030	0.024745	0.009503
Percentage points of Watson statistic [13]	0.131	0.152	0.187	0.222	0.267
Distribution (7)	0.150602	0.099526	0.049882	0.024998	0.010283
Simulation result	0.150357	0.099479	0.049745	0.024865	0.010305
Distribution (9)	0.149243	0.098704	0.050171	0.025747	0.011149

KUIPER TEST

Kuiper [1] proposed an extended Kolmogorov test statistic for testing the hypothesis that a random sample fits a law with a continuous distribution function $F(x,\theta)$. The statistic V_n for the test is given by

$$V_n = \sup_{-\infty < x < \infty} \{F_n(x) - F(x,\theta)\} - \inf_{-\infty < x < \infty} \{F_n(x) - F(x,\theta)\}$$

(here $F_n(x)$ is an empirical distribution function) and is used in the form

$$V_n = D_n^+ + D_n^-, \tag{1}$$

where $D_n^+ = \max\{i/n - F(x_i,\theta)\}$, $D_n^- = \max\{F(x_i,\theta) - (i-1)/n\}$, $i = \overline{1,n}$, n is the sample size x_i are the elements of a variational series constructed from the sample (in increasing order).

A significant disadvantage of the test employing statistic (1) is the strong dependence of the Kuiper statistic distribution $G(V_n \mid H_0)$ on the sample size n. Tables of percentage points for testing simple hypotheses using the test statistic $\sqrt{n}V_n$ can be found in [8, 9]. As the limiting distribution $G(\sqrt{n}V_n \mid H_0)$ for the statistic $\sqrt{n}V_n$ [9], Kuiper [1] gives the following distribution function:

$$G(s \mid H_0) = 1 - \sum_{m=1}^{\infty} 2(4m^2s^2 - 1)\mathrm{e}^{-2m^2s^2}. \tag{2}$$

For the modified statistic

$$V = V_n\Big(\sqrt{n} + 0.155 + \frac{0.24}{\sqrt{n}}\Big), \tag{3}$$

whose distribution is less dependent on n than the distribution of $\sqrt{n}V_n$, percentage points are presented in [10] (shown in row 2 of Table 1). The dependence of the distribution of the statistic (3) on the sample size can be neglected for $n \geq 20$ since the deviation of the real distribution of the statistic from the limiting distribution is insignificant and has little effect on the statistical inference.

In the Kuiper test, it is possible to use a statistic of the form

$$V_n^{\text{mod}} = \sqrt{n}(D_n^+ + D_n^-) + \frac{1}{3\sqrt{n}}, \tag{4}$$

where the idea of using the correction follows from the expression for the statistic of the Smirnov goodness-of-fit test [11, p. 81]. The dependence of the distribution statistics (4) on the sample size is practically negligible when $n \geq 30$.

Statistics (3) and (4) have the same limiting distributions. For small n, the difference between the distributions of statistics (3) and (4) is rather large. However, for $n \geq 20$ in the region of decision making (for values of the statistic distribution functions $G(V \mid H_0) > 0.9$ and $G(V_n^{\mathrm{mod}} \mid H_0) > 0.9$), these distributions are practically identical.

As a model of the limiting law, we can use the beta distribution of the third kind with density

$$f(x) = \frac{\theta_2^{\theta_0}}{\theta_3 B(\theta_0, \theta_1)} \frac{((x-\theta_4)/\theta_3)^{\theta_0 - 1}(1-(x-\theta_4)/\theta_3)^{\theta_1 - 1}}{[1+(\theta_2 - 1)((x-\theta_4)/\theta_3)]^{\theta_0 + \theta_1}} \tag{5}$$

($B(\cdot)$ is the beta function) and the parameter vector $\boldsymbol{\theta} = (7.8624,\ 7.6629,\ 2.6927,\ 2.6373,\ 0.495)^T$ constructed from the results of simulation of statistics (4). This model describes the distribution of statistic (4) in the whole domain of definition and, along with the limiting distribution, can be used to calculate the achieved significance level $P\{S > S^* \mid H_0\}$, i.e., the probability that in the case of validity of the tested hypothesis H_0, the statistic S of the test exceeds S^*, where S^* is the value of the statistic calculated from the sample.

WATSON TEST

The Watson test statistic [2, 3] has the form

$$U_n^2 = n \int\limits_{-\infty}^{\infty} \Big\{F_n(x) - F(x,\theta) - \int\limits_{-\infty}^{\infty} (F_n(y) - F(y,\theta))dF(y,\theta)\Big\}^2 dF(x,\theta)$$

and is used in the following form convenient for calculations:

$$U_n^2 = \sum_{i=1}^{n} \Big(F(x_i,\theta) - \frac{i-1/2}{n}\Big)^2 - n\Big(\frac{1}{n}\sum_{i=1}^{n} F(x_i,\theta) - \frac{1}{2}\Big)^2 + \frac{1}{12n}. \tag{6}$$

In testing a simple hypothesis, the percentage points of the statistic U_n^2 can be found in [3, 12]. The limiting distribution $G(U_n^2 \mid H_0)$ of the statistic U_n^2 is given in [2, 3] in the form

$$G(s \mid H_0) = 1 - 2\sum_{m=1}^{\infty} (-1)^{m-1} \mathrm{e}^{-2m^2\pi^2 s}. \tag{7}$$

Modifications of the Kuiper and Watson tests were considered in [13], and those of the Watson test in [14]. The asymptotic efficiency of the Watson test was studied in [15].

Percentage points for distributions of the modified statistics are given in [13]. In particular, the upper percentage points for the modified Watson statistics in the form

$$U_n^{2*} = (U_n^2 - 0.1/n + 0.1/n^2)(1 + 0.8/n) \tag{8}$$

take the values [13] given in row 6 of Table 1. For sample size $n \geq 20$, the difference between the distribution of statistic (8) and the limiting distribution can be neglected.

Nearly the same values of the upper percentage points are used for the distribution of statistic (6). It should be emphasized that the dependence of the distribution of statistic (6) on the sample size is insignificant.

The limiting distribution of statistic (6) for the entire domain of definition is well approximated by the inverse Gaussian model with density

$$f(x) = \frac{1}{\theta_2}\Big(\frac{\theta_0}{2\pi((x-\theta_3)/\theta_2)^3}\Big)^{1/2} \exp\Big(-\frac{\theta_0((x-\theta_3)/\theta_2 - \theta_1)^2}{2\theta_1^2((x-\theta_3)/\theta_2)}\Big) \tag{9}$$

and the vector of parameters $\boldsymbol{\theta} = (0.2044,\ 0.08344,\ 1.0,\ 0.0)^\top$, constructed from the simulation results of the empirical distribution of statistic (6). In testing simple hypotheses using the Watson test, this distribution and the limiting distribution can be used to calculate the achieved significance level.

Values of the probabilities $P(S > S_\alpha \mid H_0)$ corresponding to the given percentage points (critical values) for the Kuiper test [10] calculated using expression (2), the limiting law model (5), and the results of statistical simulation of values of statistic (4) $N = 1.7 \cdot 10^6$ are presented in Table 1 for specified significance levels α. The table also gives similar probabilities for the Watson test [13] calculated using relation (7), the limiting law model (9), and the results of statistical modeling of the distribution of statistic (6). These data allow one, on the one hand, judge the accuracy of modeling distributions of test statistics and, on the other hand, to determine the possibility of constructing good models for unknown limiting (and unsaturated) distributions of statistics that provide a fairly accurate evaluation of the achieved significance level.

ZHANG TESTS

Zhang [4–7] proposed nonparametric goodness-of-fit tests whose statistics have the following form:

$$Z_K = \max_{1 \le i \le n} \Big((i - 1/2) \log \Big\{ \frac{i - 1/2}{nF(x_i, \theta)} \Big\} + (n - i + 1/2) \log \Big[\frac{n - i + 1/2}{n\{1 - F(x_i, \theta)\}} \Big] \Big), \tag{10}$$

$$Z_A = -\sum_{i=1}^{n} \Big[\frac{\log\{F(x_i, \theta)\}}{n - i + 1/2} + \frac{\log\{1 - F(x_i, \theta)\}}{i - 1/2} \Big], \tag{11}$$

$$Z_C = \sum_{i=1}^{n} \Big[\log \Big\{ \frac{[F(x_i, \theta)]^{-1} - 1}{(n - 1/2)/(i - 3/4) - 1} \Big\} \Big]^2. \tag{12}$$

Our study [16] confirmed the validity of Zhang's statement that the power of the tests is higher than that of the Kolmogorov, Cramer–von Mises–Smirnov, and Anderson–Darling tests. However, tests with statistics (10)–(12) cannot be recommended for wide use because of the strong dependence of distributions of the statistics on the sample size n. This dependence complicates the use of the tests. Naturally, the dependence on n holds in testing composite hypotheses.

ANALYSIS OF THE POWER OF TESTS

To investigate the distributions of statistics in the case of validity of the tested $G(S \mid H_0)$ and competing $G(S \mid H_1)$ hypotheses and to estimate the power, we used the same approach being developed [17] based on computer technologies and statistical modeling. The results of statistical modeling provided accuracy of constructing statistic distributions $G(S \mid H_i)$, $i = \overline{0, 1}$ of the order of $\pm 10^{-3}$ with a confidence probability of 0.9. This value determines the maximum length of the median confidence interval that covers the true value of the distribution function at a point. To compare the powers of the investigated tests and other goodness-of-fit tests, the results of the studies are demonstrated using two pairs of the same competing laws as in [18–20].

The first pair consisted of the normal and logistic laws: the tested hypothesis H_0 corresponded to the normal law with density

$$f(x) = \frac{1}{\theta_0 \sqrt{2\pi}} \exp\Big\{ -\frac{(x - \theta_1)^2}{2\theta_0^2} \Big\},$$

and the competing hypothesis H_1 to the logistic law with density function

$$f(x) = \frac{\pi}{\theta_0 \sqrt{3}} \exp\Big\{ -\frac{\pi(x - \theta_1)}{\theta_0 \sqrt{3}} \Big\} \Big/ \Big[1 + \exp\Big\{ -\frac{\pi(x - \theta_1)}{\theta_0 \sqrt{3}} \Big\} \Big]^2$$

and parameters $\theta_0 = 1$ and $\theta_1 = 0$. In the case of a simple hypothesis H_0, the parameters of the normal law have the same values. These two laws are close and difficult to distinguish using goodness-of-fit tests.

The second pair consisted of H_0 — the Weibull distribution with density

$$f(x) = \frac{\theta_0 (x - \theta_2)^{\theta_0 - 1}}{\theta_1^{\theta_0}} \exp\Big\{ -\Big(\frac{x - \theta_2}{\theta_1} \Big)^{\theta_0} \Big\}$$

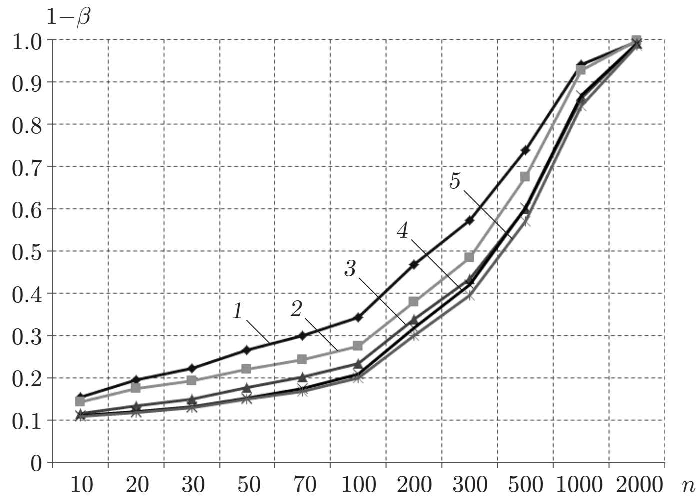

Fig. 1.

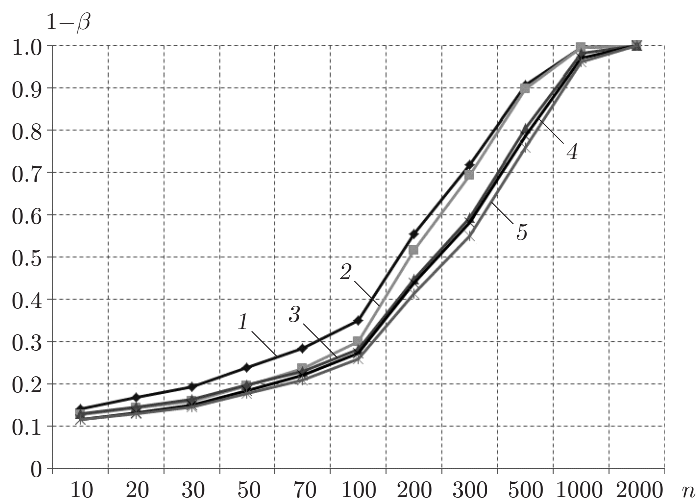

Fig. 2.

and parameters $\theta_0 = 2$ and $\theta_1 = 2$, $\theta_2 = 0$ and H_1 — the gamma distribution with density

$$f(x) = \frac{1}{\theta_1 \Gamma(\theta_0)} \Big(\frac{x - \theta_2}{\theta_1}\Big)^{\theta_0 - 1} \mathrm{e}^{-(x - \theta_2)/\theta_1}$$

and parameters $\theta_0 = 3.12154$, $\theta_1 = 0.557706$, and $\theta_2 = 0$, for which the gamma distribution is closest to the Weibull law.

The power was explored in testing simple and composite hypotheses H_0 against the simple competing hypotheses H_1 for different values of the error probability of the first kind α and different sample sizes n.

As an example, Fig. 1 shows curves of the power of the tests versus sample size n in testing the simple hypothesis H_0 (normal distribution) against the hypothesis H_1 (logistic distribution) for an error probability of the 1st kind $\alpha = 0.1$.

In this and subsequent figures, the following notation is used: curve *1* shows the dependence of the power on sample size for the Zhang test with statistic Z_C (12); *2* for the Zhang test with statistic Z_A (11); *3* for the Zhang test with statistic Z_K (10); *4* for the Watson test with statistic U_n^2 (6); *5* for the Kuiper test with statistic V_n^{mod} (5). Figure 2 shows curves of the power in testing the simple hypothesis H_0 (Weibull distribution with parameters 2, 2, and 0) against the hypothesis H_1 (gamma distribution with parameters 3.12154, 0.557706, and 0).

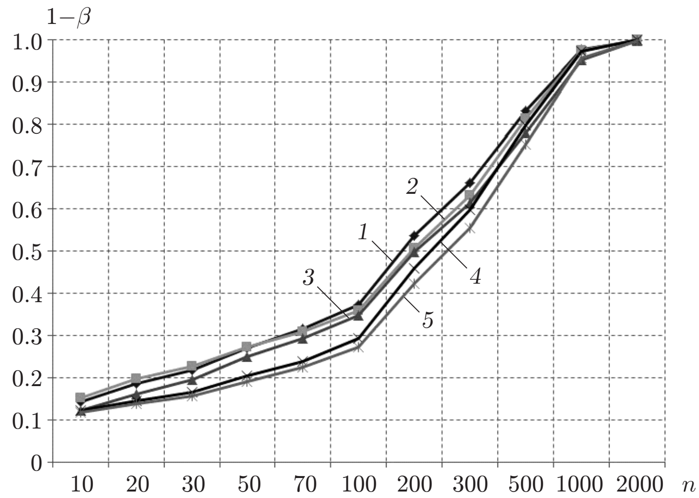

Fig. 3.

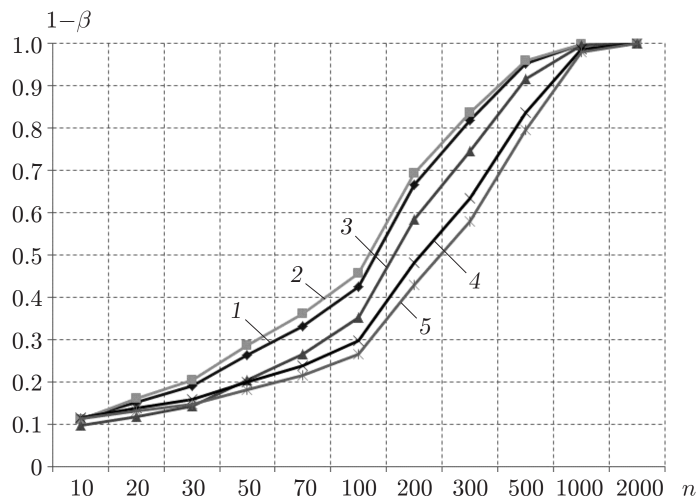

Fig. 4.

In testing composite hypotheses, the parameters of the law were estimated using the maximum likelihood method. Corresponding dependences for the same pairs of competing laws in testing composite hypotheses are presented in Figs. 3 and 4.

Comparing the estimated the powers of the tests considered and the results for the Kolmogorov (K), Cramer–von Mises–Smirnov (KMS), and Anderson–Darling (AD) tests given in [19, 20], we can order the tests according to the power as follows:

— for testing simple hypotheses for the pair of the normal law and logistic law: $Z_C \succ Z_A \succ Z_K \succ U_n^2 \succ V_n \succ AD \succ K \succ KMS$;

— for testing simple hypotheses for the pair of the Weibull law and gamma distribution: $Z_C \succ Z_A \succ Z_K \succ U_n^2 \succ V_n \succ AD \succ KMS \succ K$;

— for testing composite hypotheses for the pair of the normal law and logistic law: $Z_A \approx Z_C \succ Z_K \succ AD \succ KMS \succ U_n^2 \succ V_n \succ K$;

— for testing composite hypotheses for the pair of the Weibull law and gamma distribution: $Z_A \succ Z_C \succ AD \succ Z_K \succ KMS \succ U_n^2 \succ V_n \succ K$.

If we compare the obtained results with the power of χ^2 type tests, it turns out that for testing simple hypotheses [19], the Pearson χ^2 test is the third provided that asymptotically optimal grouping is used [17] and that the number of intervals is chosen so that the test has the maximum power [17, 21]. In the case

Table 2. Distribution laws of random variables

Name of law	Density function $f(x,\theta)$	Name of law	Density function $f(x,\theta)$
Exponential	$\frac{1}{\theta_0}\,\mathrm{e}^{-x/\theta_0}$	Laplace	$\frac{1}{2\theta_0}\,\mathrm{e}^{-\lvert x-\theta_1\rvert/\theta_0}$
Seminormal	$\frac{2}{\theta_0\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2\theta_0^2}$	Normal (Gauss)	$\frac{1}{\theta_0\sqrt{2\pi}}\,\mathrm{e}^{-(x-\theta_1)^2/2\theta_0^2}$
Rayleigh	$\frac{x}{\theta_0^2}\,\mathrm{e}^{-x^2/2\theta_0^2}$	Logarithmic normal	$\frac{1}{x\theta_0\sqrt{2\pi}}\,\mathrm{e}^{-(\ln x-\theta_1)^2/2\theta_0^2}$
Maxwell	$\frac{2x^2}{\theta_0^3\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2\theta_0^2}$	Cauchy	$\frac{\theta_0}{\pi[\theta_0^2+(x-\theta_1)^2]}$

Name of law	Density function $f(x,\theta)$
Logistic	$\frac{\pi}{\theta_0\sqrt{3}}\exp\left\{-\frac{\pi(x-\theta_1)}{\theta_0\sqrt{3}}\right\}\Big/\left[1+\exp\left\{-\frac{\pi(x-\theta_1)}{\theta_0\sqrt{3}}\right\}\right]^2$
Extreme value (maximum)	$\frac{1}{\theta_0}\exp\left\{-\frac{x-\theta_1}{\theta_0}-\exp\left(-\frac{x-\theta_1}{\theta_0}\right)\right\}$
Extreme value (minimum)	$\frac{1}{\theta_0}\exp\left\{\frac{x-\theta_1}{\theta_0}-\exp\left(\frac{x-\theta_1}{\theta_0}\right)\right\}$
Weibull	$\frac{\theta_0 x^{\theta_0-1}}{\theta_1^{\theta_0}}\exp\{-(x/\theta_1)^{\theta_0}\}$
Sb-Johnson ($Sb(\theta_0,\theta_1,\theta_2,\theta_3)$)	$\frac{\theta_1\theta_2}{(x-\theta_3)(\theta_2+\theta_3-x)}\exp\left\{-\frac{1}{2}\left[\theta_0-\theta_1\ln\frac{x-\theta_3}{\theta_2+\theta_3-x}\right]^2\right\}$
Sl-Johnson ($Sl(\theta_0,\theta_1,\theta_2,\theta_3)$)	$\frac{\theta_1}{(x-\theta_3)\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left[\theta_0+\theta_1\ln\frac{x-\theta_3}{\theta_2}\right]^2\right\}$
Su-Johnson ($Su(\theta_0,\theta_1,\theta_2,\theta_3)$)	$\frac{\theta_1}{\sqrt{2\pi}\sqrt{(x-\theta_3)^2+\theta_2^2}}\exp\left\{-\frac{1}{2}\left[\theta_0+\theta_1\ln\left\{\frac{x-\theta_3}{\theta_2}+\sqrt{\left(\frac{x-\theta_3}{\theta_2}\right)^2+1}\right\}\right]^2\right\}$

of testing composite hypotheses [20], the positions of the Pearson χ^2 tests and the Nikulin–Rao–Robson χ^2 tests [22–24] worsen: they are the seventh and eighth in the decreasing series of the power of tests. However, we note that the powers of these tests can be maximized with respect to the given competing hypotheses through the optimal choice of the boundaries and number of grouping intervals [17, 21].

USE OF THE KUIPER AND WATSON TESTS IN TESTING COMPOSITE HYPOTHESES

As mentioned above for testing composite hypotheses of the form H_0: $F(x) \in \{F(x,\theta),\ \theta \in \Theta\}$, if the estimate $\hat{\theta}$ of the scalar or vector distribution parameter $F(x,\theta)$ is calculated from the same sample, nonparametric goodness-of-fit tests lose the distribution free property [25].

The problem of testing composite hypotheses using nonparametric goodness-of-fit tests has been solved using different approaches [10, 13, 26–30]. In [31, 32], we employed numerical methods and statistical modeling. The results obtained were used to develop recommendations for the use of the nonparametric Kolmogorov, Cramer–von Mises–Smirnov, and Anderson–Darling tests [33, 34]. These results were later refined and expanded [35–44] and are most fully presented in [17].

In this paper, we present tables of percentage points and distribution models for the Kuiper and Watson test statistics that are recommended to be used in testing composite hypotheses for some parametric distribution laws of random quantities widely used in applications.

Table 2 contains a list of distribution laws for which composite hypotheses can be tested using the approximations constructed in this paper for the limiting distributions of nonparametric goodness-of-fit test statistics.

The tables of percentage points and distribution models for the test statistic were constructed based on simulated samples of statistics of size $N = 1.7 \cdot 10^6$. For such N, the absolute value of the difference

Table 3. Upper percentage points and limiting distribution models for the Kuiper test statistic when using MLEs

Name of law	Estimated parameter	Percentage points 0.1	0.05	0.01	Model
Exponential, Raileygh, Maxwell	Scale	1.540	1.661	1.905	$B_3(5.5932, 7.6149, 2.1484, 2.3961, 0.5630)$
Seminormal	Scale	1.543	1.664	1.907	$B_3(11.4707, 40.7237, 7.020, 20.3675, 0.3989)$
Laplace	Scale	1.469	1.587	1.825	$B_3(7.8324, 8.3778, 2.6906, 2.4820, 0.4830)$
	Location	1.473	1.597	1.850	$B_3(9.1630, 6.6097, 4.0210, 2.4081, 0.4900)$
	Both parameters	1.278	1.365	1.541	$B_3(10.0376, 7.8452, 3.4694, 1.9586, 0.4756)$
Normal, lognormal	Scale	1.494	1.611	1.847	$B_3(6.3057, 8.1797, 2.3279, 2.4413, 0.5370)$
	Location	1.540	1.662	1.908	$B_3(5.5932, 7.6149, 2.1484, 2.3961, 0.5630)$
	Both parameters	1.402	1.505	1.709	$B_3(7.4917, 8.0016, 2.4595, 2.1431, 0.4937)$
Cauchy	Scale or location	1.435	1.560	1.815	$B_3(3.8425, 5.9345, 2.4284, 2.1927, 0.6150)$
	Both parameters	1.126	1.197	1.337	$B_3(9.4267, 7.5349, 3.2515, 1.5491, 0.4700)$
Logistic	Scale	1.470	1.588	1.826	$B_3(9.7224, 7.8186, 3.2399, 2.4541, 0.4370)$
	Shifr	1.511	1.633	1.880	$B_3(9.1363, 6.9693, 3.4630, 2.3985, 0.4790)$
	Both parameters	1.337	1.432	1.622	$B_3(14.3460, 18.6137, 3.6366, 3.9560, 0.3525)$
Extreme values, Weibull	Scale*	1.504	1.622	1.861	$Sl(1.2459, 4.0123, 1.3063, 0.1873)$
	Location**	1.540	1.662	1.908	$B_3(5.5932, 7.6149, 2.1484, 2.3961, 0.5630)$
	Both parameters	1.411	1.516	1.726	$Sl(1.4012, 5.0846, 1.4465, -0.0070)$

Note: * when estimating the shape parameter of the Weibull distribution, ** when estimating the scale parameter of the Weibull distribution.

between the true law $G(S \mid H_0)$ of statistic distribution and the modeled empirical law $G_N(S \mid H_0)$ does not exceed 10^{-3}. Values of the test statistics were calculated from samples of pseudorandom quantities of size $n = 10^3$ generated in accordance with the observed law $F(x, \theta)$. In this situation, the distribution $G(S_n \mid H_0)$ practically coincides with the limiting distribution $G(S \mid H_0)$. In statistical analysis, the models presented in this paper can be used starting from sample sizes $n > 25$.

The distribution $G(S \mid H_0)$ of the Kuiper and Watson test statistics are best approximated by the family of beta distributions of the third kind $B_3(\theta_0, \theta_1, \theta_2, \theta_3, \theta_4)$, whose density function is defined by relation (5), and the Sl-Johnson distribution family ($Sl(\theta_0, \theta_1, \theta_2, \theta_3)$) (Table 2).

The upper percentage points and the constructed limiting distribution models for the Kuiper test statistic using maximum likelihood estimates (MLE) are presented in Table 3 for the following laws: exponential, seminormal, Rayleigh, Maxwell, Laplace, normal, lognormal, Cauchy, logistic, extreme values (maximum and minimum), and Weibull. For the same distribution laws, the upper percentage points and the constructed distribution models for the Watson test statistic are given in Table 4.

The upper percentage points and models for the limiting distribution of statistics for testing composite hypotheses for the Sb-Johnson distribution laws (using MLEs) are presented in Table 5, for the Sl-Johnson laws in Table 6, and for Su-Johnson laws in Table 7.

The accuracy of the constructed statistic distribution models for testing composite hypotheses and the possibility of using these models in statistical analysis may be demonstrated as follows.

Table 8 illustrates the results of constructing statistic distribution models for testing composite hypotheses on goodness of fit of the analyzed sample to the logistic law with calculation of the MLEs of the location and scale parameters of this law from the sample. The constructed models are given for the Kuiper test in Table 3 and for the Watson test in Table 4.

The column $F_N^1(v)$ in Table 8 presents estimates of the Kuiper statistic distribution function corresponding to the values of statistic v and obtained from the simulated empirical statistic distribution for which the model $B_3(14.3460, 18.6137, 3.6366, 3.9560, 0.3525)$ in Table 3 was constructed. The respective values of the

Table 4. Upper percentage points and limiting distribution models for the Watson test statistic when using MLEs

Name of law	Estimated paremeter	Percentage points			Model
		0.1	0.05	0.01	
Exponential, Raileygh, Maxwell	Scale	0.129	0.159	0.230	$B_3(4.0419, 2.9119, 10.5931, 0.5000, 0.0096)$
Seminormal	Scale	0.131	0.161	0.232	$B_3(4.9988, 3.8721, 15.1781, 0.6900, 0.0059)$
Laplace	Scale	0.115	0.144	0.214	$B_3(9.2136, 3.8610, 30.5491, 0.7010, 0.0015)$
	Location	0.111	0.139	0.209	$B_3(7.4479, 3.2650, 30.7784, 0.6227, 0.0063)$
	Both parameters	0.071	0.084	0.114	$B_3(9.0116, 5.3554, 17.3201, 0.3908, 0.0038)$
Normal, Logarithmic Normal	Scale	0.122	0.151	0.221	$B_3(8.8122, 3.7536, 29.8074, 0.7171, 0.0019)$
	Location	0.127	0.157	0.228	$B_3(3.6769, 4.4438, 9.8994, 0.6805, 0.0082)$
	Both parameters	0.096	0.116	0.164	$B_3(3.5230, 4.4077, 9.2281, 0.4785, 0.0104)$
Cauchy	Scale or location	0.105	0.133	0.203	$Sl(2.7778, 1.5065, 0.2690, 0.0049)$
	Both parameters	0.052	0.061	0.081	$B_3(8.3558, 4.8650, 12.0768, 0.1930, 0.0049)$
Logistic	Scale	0.115	0.144	0.214	$B_3(9.2136, 3.8610, 30.5491, 0.7010, 0.0015)$
	Location	0.119	0.148	0.218	$B_3(3.9730, 3.9414, 13.2655, 0.6637, 0.0090)$
	Both parameters	0.081	0.098	0.135	$B_3(4.2608, 4.6784, 9.3054, 0.3810, 0.0084)$
Extreme values, Weibull	Scale*	0.122	0.151	0.221	$B_3(8.8122, 3.7536, 29.8074, 0.7171, 0.0019)$
	Location**	0.129	0.159	0.230	$B_3(4.9988, 3.8721, 15.1781, 0.6792, 0.0061)$
	Both parameters	0.097	0.118	0.165	$Sl(1.2863, 1.6736, 0.0927, 0.0052)$

Note: $*$ when estimating the shape parameter of the Weibull distribution, $**$ when estimating the scale parameter of the Weibull distribution.

Table 5. Upper percentage points and limiting distribution models for nonparametric goodness-of-fit test statistics in the case of testing hypotheses about Sb-Johnson distributions using MLEs

Estimated paremeter	Percentage points			Model
	0.1	0.05	0.01	
for the Kuiper test				
θ_0	1.540	1.662	1.908	$B_3(5.5932, 7.6149, 2.1484, 2.3961, 0.5630)$
θ_1	1.494	1.611	1.847	$B_3(6.3057, 8.1797, 2.3279, 2.4413, 0.5370)$
θ_0, θ_1	1.402	1.505	1.709	$B_3(7.4917, 8.0016, 2.4595, 2.1431, 0.4937)$
for the Watson test				
θ_0	0.127	0.157	0.228	$B_3(3.6769, 4.4438, 9.8994, 0.6805, 0.0082)$
θ_1	0.122	0.151	0.221	$B_3(8.8122, 3.7536, 29.8074, 0.7171, 0.0019)$
θ_0, θ_1	0.096	0.116	0.164	$B_3(3.5230, 4.4077, 9.2281, 0.4785, 0.0104)$

statistic distribution function calculated for this model are given in the column $F(v)$. The column $F_N^2(v)$ contains estimates of the distribution function obtained from the control (newly generated) sample of statistics.

Similarly, the corresponding estimates for the distribution function of the Watson test statistics in constructing the model $B_3(4.2608, 4.6784, 9.3054, 0.3810, 0.0084)$ from Table 4 are shown in the columns $F_N^1(u_n^2)$, $F(u_n^2)$, and $F_N^2(u_n^2)$ in Table 8.

As seen from Table 8, for the Kuiper test, $\max|F_N^1(v) - F_N^2(v)| = 0.000522$, which fits in a 90% confidence interval, whose maximum value for $N = 1.7 \cdot 10^6$ does not exceed 0.002. The deviations of the empirical distributions from the constructed model are $\max|F_N^1(v) - F(v)| = 0.001509$ and $\max|F(v) - F_N^2(v)| = 0.001579$. Similarly, for the Watson test, we have $\max|F_N^1(u_n^2) - F_N^2(u_n^2)| = 0.000537$, $\max|F_N^1(u_n^2) - F(u_n^2)| = 0.000932$, and $\max|F(u_n^2) - F_N^2(u_n^2)| = 0.001323$.

Table 6. Upper percentage points and limiting distribution models for nonparametric goodness-of-fit test statistics in the case of testing hypotheses about Sl-Johnson distributions using MLEs

Estimated parameter	Percentage points			Model
	0.1	0.05	0.01	
for the Kuiper test				
θ_0	1.540	1.662	1.908	$B_3(5.5932, 7.6149, 2.1484, 2.3961, 0.5630)$
θ_1	1.512	1.631	1.872	$B_3(6.7423, 8.0549, 2.4935, 2.4976, 0.5250)$
θ_2	1.540	1.662	1.908	$B_3(5.5932, 7.6149, 2.1484, 2.3961, 0.5630)$
θ_0, θ_1	1.402	1.505	1.709	$B_3(7.4917, 8.0016, 2.4595, 2.1431, 0.4937)$
θ_0, θ_2	1.540	1.662	1.908	$B_3(5.5932, 7.6149, 2.1484, 2.3961, 0.5630)$
θ_1, θ_2	1.402	1.505	1.709	$B_3(7.4917, 8.0016, 2.4595, 2.1431, 0.4937)$
$\theta_0, \theta_1, \theta_2$	1.402	1.505	1.709	$B_3(7.4917, 8.0016, 2.4595, 2.1431, 0.4937)$
for the Watson test				
θ_0	0.127	0.157	0.228	$B_3(3.6769, 4.4438, 9.8994, 0.6805, 0.0082)$
θ_1	0.124	0.153	0.223	$B_3(3.4122, 4.9262, 9.6902, 0.7643, 0.0087)$
θ_2	0.127	0.157	0.228	$B_3(3.6769, 4.4438, 9.8994, 0.6805, 0.0082)$
θ_0, θ_1	0.096	0.116	0.164	$B_3(3.5230, 4.4077, 9.2281, 0.4785, 0.0104)$
θ_0, θ_2	0.127	0.157	0.228	$B_3(3.6769, 4.4438, 9.8994, 0.6805, 0.0082)$
θ_1, θ_2	0.096	0.116	0.164	$B_3(3.5230, 4.4077, 9.2281, 0.4785, 0.0104)$
$\theta_0, \theta_1, \theta_2$	0.096	0.116	0.164	$B_3(3.5230, 4.4077, 9.2281, 0.4785, 0.0104)$

INTERACTIVE MODE OF STUDYING STATISTIC DISTRIBUTIONS

The major factor hindering the use of nonparametric goodness-of-fit tests to test composite hypotheses for a wide range of possible parametric distribution laws in various applications for describing the observed random values (measurement errors) is the dependence of distributions of test statistics on particular values of the shape parameter (or parameters) of the law corresponding to the hypothesis being tested (in the case of families of gamma and beta distributions, generalized Weibull, inverse Gaussian, etc.). As a rule, this refers to laws that are most promising in various applications, in the analysis of the survival and reliability of complex products and systems.

Since estimates of the parameters become known only in the process of analysis, the statistic distribution required to test the hypothesis cannot be found in advance (before calculating estimates from the analyzed sample!). In the case of tests with statistics (10)–(12), the problem is exacerbated by the dependence of the statistic distributions on the sample size. It follows that the distributions of the statistics of the tests used should be found interactively during ongoing statistical analysis [45] and then used in the inference from the results of the test of the composite hypothesis.

Implementation of this interactive mode requires advanced software that allows (as in [46]) parallelizing the simulation processes and invoking available computing resources. In the case of parallelizing, the time of constructing the distribution $G_N(S_n \,|\, H_0)$ of the test statistic (with the required accuracy) necessary for testing the hypothesis and determining the achieved significance level $P\{S_n \geq S^*\}$, where S^* is the value of the statistic calculated from the sample is negligible compared to the time required for complete solution of the problem of statistical analysis.

In [46], the interactive mode of studying statistic distributions was implemented for the following nonparametric goodness-of-fit tests: Kolmogorov, Cramer–von Mises–Smirnov, Anderson–Darling, Kuiper, Watson, and Zhang (three tests). In this case, different methods can be used to estimate the parameters.

In survival and reliability analysis of complex systems, samples are usually characterized by the presence of censored and randomly (repeatedly) censored observations. The adequacy of the reliability functions constructed are also verified using goodness-of-fit tests, but, in this case, they are applied to the generated samples of residues. In the case of censored samples, the distributions of the statistics of the employed modified nonparametric goodness-of-fit tests depend, in addition, on the distribution laws of the censoring moments and the degree of censoring of the samples [47–49]. Under such conditions, the interactive mode of research is particularly needed.

The following examples demonstrates the accuracy the determination of the achieved significance level depending on the sample size N of the interactively modeled empirical distribution of the statistic [46].

Table 7. Upper percentage points and limiting distribution models for nonparametric goodness-of-fit test statistics in the case of testing hypotheses about Su-Johnson distributions using MLEs

Estimated parameter	Percentage points			Model
	0.1	0.05	0.01	
for the Kuiper test				
θ_0	1.540	1.662	1.908	$B_3(5.5932, 7.6149, 2.1484, 2.3961, 0.5630)$
θ_1	1.512	1.631	1.872	$B_3(6.7676, 8.3605, 2.3501, 2.4976, 0.5142)$
θ_2	1.491	1.612	1.857	$B_3(7.5884, 8.1397, 2.6781, 2.4982, 0.4882)$
θ_3	1.517	1.638	1.885	$B_3(8.1449, 7.2651, 3.0338, 2.4418, 0.4880)$
θ_0, θ_1	1.402	1.505	1.709	$B_3(8.1449, 7.2650, 3.0338, 2.1431, 0.5015)$
θ_0, θ_2	1.393	1.496	1.703	$B_3(7.5234, 7.3134, 2.7694, 2.1076, 0.5035)$
θ_0, θ_3	1.390	1.496	1.713	$B_3(8.0187, 7.7542, 2.7862, 2.1751, 0.4800)$
θ_1, θ_2	1.414	1.525	1.749	$B_3(8.6702, 7.5387, 2.9284, 2.2036, 0.4600)$
θ_1, θ_3	1.375	1.475	1.675	$B_3(8.6702, 7.5387, 2.9284, 2.0887, 0.4740)$
θ_2, θ_3	1.350	1.447	1.640	$B_3(9.0132, 7.9999, 2.8585, 2.0644, 0.4635)$
$\theta_0, \theta_1, \theta_2$	1.324	1.422	1.621	$B_3(10.7806, 8.4043, 3.2432, 2.1461, 0.4150)$
$\theta_0, \theta_1, \theta_3$	1.333	1.431	1.629	$B_3(10.3455, 8.0495, 3.5687, 2.1993, 0.4463)$
$\theta_0, \theta_2, \theta_3$	1.296	1.388	1.575	$B_3(10.3223, 7.7893, 3.3393, 2.0021, 0.4358)$
$\theta_1, \theta_2, \theta_3$	1.299	1.394	1.584	$B_3(10.5957, 8.2600, 3.2334, 2.0676, 0.4194)$
$\theta_0, \theta_1, \theta_2, \theta_3$	1.235	1.321	1.494	$B_3(9.9689, 7.3418, 3.4037, 1.8225, 0.4438)$
for the Watson test				
θ_0	0.127	0.157	0.228	$B_3(3.6769, 4.4438, 9.8994, 0.6805, 0.0082)$
θ_1	0.124	0.153	0.223	$B_3(3.4122, 4.9262, 9.6902, 0.7643, 0.0087)$
θ_2	0.117	0.146	0.215	$B_3(6.0296, 3.7175, 22.6978, 0.7115, 0.0057)$
θ_3	0.121	0.150	0.220	$B_3(7.4154, 3.9208, 22.4649, 0.6800, 0.0022)$
θ_0, θ_1	0.096	0.116	0.164	$B_3(3.5230, 4.4077, 9.2281, 0.4785, 0.0104)$
θ_0, θ_2	0.093	0.114	0.161	$B_3(4.0651, 4.8643, 9.5614, 0.4903, 0.0078)$
θ_0, θ_3	0.092	0.113	0.162	$B_3(4.4170, 4.9456, 10.4292, 0.5005, 0.0067)$
θ_1, θ_2	0.099	0.123	0.181	$B_3(5.5181, 4.1815, 16.0852, 0.5478, 0.0055)$
θ_1, θ_3	0.089	0.108	0.151	$B_3(5.7461, 4.4051, 13.9768, 0.4528, 0.0060)$
θ_2, θ_3	0.084	0.101	0.141	$B_3(5.9952, 4.3409, 13.8757, 0.4020, 0.0060)$
$\theta_0, \theta_1, \theta_2$	0.077	0.093	0.131	$B_3(5.5809, 4.9570, 14.1052, 0.4540, 0.0060)$
$\theta_0, \theta_1, \theta_3$	0.080	0.097	0.137	$B_3(5.8959, 4.4478, 14.5923, 0.4132, 0.0060)$
$\theta_0, \theta_2, \theta_3$	0.072	0.087	0.121	$B_3(6.1780, 4.6712, 14.5568, 0.3791, 0.0060)$
$\theta_1, \theta_2, \theta_3$	0.072	0.087	0.121	$B_3(6.1780, 4.6712, 14.5568, 0.3791, 0.0060)$
$\theta_0, \theta_1, \theta_2, \theta_3$	0.062	0.074	0.101	$B_3(7.3816, 4.4215, 14.1896, 0.2616, 0.0053)$

Example. It is required to test the composite hypothesis that the following sample of size $n = 100$ fits an inverse Gaussian distribution with density (9):

0.945	1.040	0.239	0.382	0.398	0.946	1.248	1.437	0.286	0.987
2.009	0.319	0.498	0.694	0.340	1.289	0.316	1.839	0.432	0.705
0.371	0.668	0.421	1.267	0.466	0.311	0.466	0.967	1.031	0.477
0.322	1.656	1.745	0.786	0.253	1.260	0.145	3.032	0.329	0.645
0.374	0.236	2.081	1.198	0.692	0.599	0.811	0.274	1.311	0.534
1.048	1.411	1.052	1.051	4.682	0.111	1.201	0.375	0.373	3.694
0.426	0.675	3.150	0.424	1.422	3.058	1.579	0.436	1.167	0.445
0.463	0.759	1.598	2.270	0.884	0.448	0.858	0.310	0.431	0.919
0.796	0.415	0.143	0.805	0.827	0.161	8.028	0.149	2.396	2.514
1.027	0.775	0.240	2.745	0.885	0.672	0.810	0.144	0.125	1.621

The location parameter $\theta_3 = 0$ is assumed to be given.

Table 8. Distribution functions of the Kuiper and Watson test statistics in testing goodness of fit to the logistic law for the case of calculation of MLEs for two parameters of the law

For the Kuiper test				For the Watson test			
v	$F_N^1(v)$	$F(v)$	$F_N^2(v)$	u_n^2	$F_N^1(u_n^2)$	$F(u_n^2)$	$F_N^2(u_n^2)$
0.6	0.000869	0.000987	0.000848	0.012	0.001923	0.001182	0.001945
0.7	0.015031	0.015772	0.015026	0.024	0.120052	0.120838	0.119515
0.8	0.081186	0.081282	0.080664	0.036	0.376606	0.375904	0.376442
0.9	0.224404	0.223060	0.224009	0.048	0.602155	0.601577	0.601897
1.0	0.417929	0.417090	0.417482	0.060	0.755732	0.755849	0.755466
1.1	0.610865	0.611452	0.610709	0.072	0.851977	0.852258	0.851492
1.2	0.765689	0.767198	0.765619	0.084	0.910636	0.910653	0.910282
1.3	0.872154	0.872875	0.871804	0.096	0.946156	0.945746	0.946005
1.4	0.936271	0.935934	0.935922	0.108	0.967708	0.966874	0.967672
1.5	0.970832	0.969894	0.970669	0.120	0.980601	0.979669	0.980678
1.6	0.987655	0.986702	0.987742	0.132	0.988400	0.987471	0.988269
1.7	0.995212	0.994446	0.995231	0.144	0.993021	0.992259	0.993041
1.8	0.998289	0.997797	0.998248	0.156	0.995797	0.995213	0.995842
1.9	0.999419	0.999168	0.999416	0.168	0.997516	0.997044	0.997457
2.0	0.999818	0.999701	0.999798	0.180	0.998483	0.998181	0.998442

Table 9. Achieved significance levels for goodness-of-fit tests for different $\boldsymbol{N}$

Values of test statistics	$N = 10^3$	$N = 10^4$	$N = 10^5$	$N = 10^6$
$V_n^{\text{mod}} = 1.1113$	0.479	0.492	0.493	0.492
$U_n^2 = 0.05200$	0.467	0.479	0.483	0.482
$Z_A = 3.3043$	0.661	0.681	0.679	0.678
$Z_C = 4.7975$	0.751	0.776	0.777	0.776
$Z_K = 1.4164$	0.263	0.278	0.272	0.270
$K = 0.5919$	0.643	0.659	0.662	0.662
$KMS = 0.05387$	0.540	0.557	0.560	0.561
$AD = 0.3514$	0.529	0.549	0.548	0.547

The shape parameters θ_0 and θ_1 and the scale parameter θ_2 are estimated from the sample. The MLEs of the parameters found from this sample are $\hat{\theta}_0 = 0.7481$, $\hat{\theta}_1 = 0.7808$, and $\hat{\theta}_2 = 1.3202$. The distribution of the statistics of all the nonparametric goodness-of-fit tests in this case depend on the values of the shape parameters θ_0 and θ_1 [40–42], do not depend on the value of the scale parameter θ_2, and must be found for the values $\theta_0 = 0.7481$ and $\theta_1 = 0.7808$.

The values S_i^* of the Kuiper, Watson, Zhang, Kolmogorov, Cramer–von Mises–Smirnov, and Anderson–Darling test statistics calculated from the sample and the achieved significance levels corresponding to these values $P\{S \geq S_i^* \mid H_0\}$ (p-value) obtained with different modeling accuracy (for different size N of the simulated samples of statistics) are shown in Table 9.

CONCLUSIONS

The study show that in testing simple hypotheses, the Kuiper and Watson goodness-of-fit tests have an advantage in power over the Kolmogorov, Cramer–von Mises–Smirnov, and Anderson–Darling tests. use of these tests for simple hypothesis involves no difficulties.

In testing composite hypotheses, the Kuiper and Watson tests lose the available advantage in power. However, this does not imply rejection of the use of these tests since the larger the number of tests using various measures of deviation of the empirical distribution from the theoretical distribution, the higher the quality of statistical inferences. The statistic distribution models and tables of percentage points constructed in this work allow the Kuiper and Watson tests to be correctly used to test composite hypotheses for a number of parametric models of distribution laws (in the case of using MLEs).

The Zhang tests with statistics Z_C and Z_A have an undeniable advantage in power over all others. This is especially noticeable in testing simple hypotheses. Some difficulties in using the tests associated with a significant dependence of the statistic distributions on the sample size are eliminated due to the interactive mode.

Due to the interactive mode of studying statistic distributions [46], the Kolmogorov, Cramer–von Mises–Smirnov, Anderson–Darling, Kuiper, Watson, and Zhang goodness-of-fit tests (with statistics Z_C, Z_A, and Z_K) can also be correctly used in the cases where the distribution of the employed test statistic corresponding to the validity of the composite hypothesis H_0 is unknown by the time of test of this hypothesis H_0. For the Zhang tests, this mode also provides testing of simple hypotheses for arbitrary sample sizes.

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REFERENCES

1. N. H. Kuiper, "Tests Concerning Random Points on a Circle," Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen, Ser. A **63**, 38–47 (1960).
2. G. S. Watson, "Goodness-of-Fit Tests on a Circle. I," Biometrika **48** (1–2), 109–114 (1961).
3. G. S. Watson, "Goodness-of-Fit Tests on a Circle. II," Biometrika **49** (1–2), 57–63 (1962).
4. J. Zhang, *Powerful Goodness-of-Fit and Multi-Sample Tests*, PhD Thesis (York University, Toronto. 2001). http://www.collectionscanada.gc.ca/obj/s4/f2/dsk3/ftp05/NQ66371.pdf.
5. J. Zhang, "Powerful Goodness-of-Fit Tests Based on the Likelihood Ratio," J. Royal Stat. Soc. Ser. B **64** (2), 281–294 (2002).
6. J. Zhang and Yu. Wub, "Likelihood-Ratio Tests for Normality," Computat. Stat. Data Anal. **49** (3), 709–721 (2005).
7. J. Zhang, "Powerful Two-Sample Tests Based on the Likelihood Ratio," Technometrics **48** (1), 95–103 (2006).
8. H. Arsham, "Kuiper's P-Value as a Measuring Tool and Decision Procedure for the Goodness-of-Fit Test," J. Appl. Stat. **15** (2), 131–135 (1988).
9. M. A. Stephens, "The Goodness-of-Fit Statistic V_N: Distribution and Significance Points," Biometrika **52** (3–4), 309–321 (1965).
10. M. A. Stephens, "EDF Statistics for Goodness of Fit and Some Comparisons," J. Amer. Stat. Assoc. **69** (347), 730–737 (1974).
11. L. N. Bol'shev and N. V. Smirnov, *Tables of Mathematical Statistics* (Nauka, Moscow, 1983) [in Russian].
12. E. S. Pearson and H. O. Hartley, *Biometrika Tables for Statisticss* (Cambridge University Press, Cambridge, 1972). Vol. 2.
13. M. A. Stephens, "Use of the Kolmogorov–Smirnov, Cramer–von Mises and Related Statistics without Extensive Tables," J. Royal Stat. Soc. Ser. B **32** (1), 115–122 (1970).
14. J. A. Koziol, "A Modification of Watson's Statistic for Goodness-of-Fit," Commun. Stat. Theory and Methods **18** (10), 3739–3747 (1989).
15. Ya. Yu. Nikitin, "Bahadur Effectiveness of Watson — Darling Goodness-of-Fit Tests," J. Math. Sci. **43** (6), 2833–2838 (1988).
16. A. V. Kozlova and B. Yu. Lemeshko, "Distributions of the Statistics and Power of the Nonparametric Goodness-of-Fit Tests Proposed by Jin Zhang," in *Proc. Ros. Conf. on Computer Science and Telecommunications Problems*, Novosibirsk, 2007, Vol. 1, pp. 136–139.
17. B. Yu. Lemeshko, S. B. Lemeshko, S. N. Postovalov, and E. V. Chimitova, *Statistical Analysis of Data, Modeling, and Study of Probability Laws: Computer Approach* (Izd. NGTU, Novosibirsk, 2011) [in Russian].
18. B. Yu. Lemeshko, S. B. Lemeshko, and S. N. Postovalov, "Power of Goodness-of-Fit Test for Close Alternatives," Izmer. Tekhnika, No. 2, 22–27 (2007).
19. B. Yu. Lemeshko, S. B. Lemeshko, and S. N. Postovalov, "Comparative Analysis of the Power of Goodness-of-Fit Tests for Close Competing Hypotheses. I. Testing Simple Hypotheses," Sib. Zh. Industr. Mat. **11** (2(34)), 96–111 (2008).
20. B. Yu. Lemeshko, S. B. Lemeshko, and S. N. Postovalov, "Comparative Analysis of the Power of Goodness-of-Fit Tests for Close Alternatives. II. Testing Composite Hypotheses," Sib. Zhurn. Industr. Mat. **11** (4(36)), 78–93 (2008).

21. B. Yu. Lemeshko and E. V. Chimitova, "On the Choice of the Number of Intervals in χ^2 Type Goodness-of-Fit Tests," Zavodskaya Laboratoriya, Diagnostika Materialov **69** (1), 61–67 (2003).
22. M. S. Nikulin, "On the Chi-Square Test for Continuous Distributions," Teoriya Veroyatnosti i Ee Primenenie **18** (3), 675–676 (1973).
23. M. S. Nikulin, "Chi-Square Test for Continuous Distributions with Location and Scale Parameters," Teoriya Veroyatnosti i ee Primenenie" **18** (3), 583–591 (1973).
24. K. C. Rao and D. S. Robson, "A Chi-Squared Statistic for Goodness-of-Fit Tests within the Exponential Family," Commun. Stat. **3** (12), 1139–1153 (1974).
25. M. Kac, J. Kiefer, and J. Wolfowitz, "On Tests of Normality and Other Tests of Goodness-of-Fit Based on Distance Methods," Ann. Math. Stat. **26** (1), 189–211 (1955).
26. D. A. Darling, "The Cramer–Smirnov Test in the Parametric Case," Ann. Math. Stat. **26** (1), 1–20 (1955).
27. H. W. Lilliefors, "On the Kolmogorov–Smirnov Test for Normality with Mean and Variance Unknown," J. Amer. Stat. Assoc. **62** (318), 399–402 (1967).
28. J. Durbin, "Weak Convergence of the Sample Distribution Function when Parameters are Estimated," Ann. Stat. **1** (2), 279–290 (1973).
29. G. V. Martynov, *Omega Squared Tests* (Nauka, Moscow, 1978) [in Russian].
30. Yu. N. Tyurin, "On the Limiting Distribution of Kolmogorov–Smirnov Statistics for a Composite Hypothesis," Izv. Akad. Nauk SSSR. Ser. Matematicheskaya **48** (6), 1314–1343 (1984).
31. B. Yu. Lemeshko and S. N. Postovalov, "Distributions of Nonparametric Goodness-of-Fit Statistics in Estimating Observed Laws from Samples of Parameters," Zavodskaya Laboratoriya **64** (3), 61–72 (1998).
32. B. Yu. Lemeshko and S. N. Postovalov, "Application of Nonparametric Goodness-of-Fit Tests for Composite Hypotheses," Avtometriya, No. 2, 88–102 (2001).
33. B. Yu. Lemeshko and S. N. Postovalov, *Applied Statistics. Rules for Testing Goodness of Fit between Experimental and Theoretical Distributions. Guidelines. Pt. II. Nonparametric Tests* (Izd-vo NGTU, Novosibirsk, 1999) [in Russian].
34. P 50.1.037-2002. *Recommendations for Standardization. Applied Statistics. Tests of Goodness of Fit between Experimental and Theoretical Distributions. Pt. II. Nonparametric Tests* (Izd-vo Standartov, Moscow, 2002) [in Russian].
35. B. Yu. Lemeshko, S. N. Postovalov, and A. V. Frantsuzov, "Use of Nonparametric Goodness-of-Fit Tests to Test the Adequacy of Nonparametric Models," Avtometriya, No. 2, 3–14 (2002).
36. B. Yu. Lemeshko and A. A. Maklakov, "Nonparametric Tests for Testing Composite Hypotheses of Goodness-of-Fit to Exponential Distributions," Avtometriya **40** (3), 3–20 (2004).
37. S. B. Lemeshko and B. Yu. Lemeshko, "Distributions of Nonparametric Goodness-of-Fit Test Statistics in Testing Hypothesis for Beta Distributions," DAN VSh Rossii No. 2(9), 6–16 (2007).
38. B. Yu. Lemeshko and S. B. Lemeshko, " Distribution Models for Nonparametric Goodness-of-Fit Test Statistics in Testing Composite Hypotheses using Maximum Likelihood Estimates. Pt. I," Izmer. Tekhnika, No. 6, 3–11 (2009).
39. B. Yu. Lemeshko and S. B. Lemeshko, "Distribution Models for Nonparametric Goodness-of-Fit Test Statistics in Testing Composite Hypotheses using Maximum Likelihood Estimates. Pt. II," Izmer. Tekhnika, No. 8, 17–26 (2009).
40. B. Yu. Lemeshko, S. B. Lemeshko, M. S. Nikulin, and N. Saaidia, "Modeling Distributions of Nonparametric Goodness-of-Fit Statistics in Testing Composite Hypotheses for the Inverse Gaussian Law," AIT, No. 7, 83–102 (2010).
41. B. Yu. Lemeshko, S. B. Lemeshko, and S. N. Postovalov, "Statistic Distribution Models for Some Nonparametric Goodness-of-Fit Tests in Testing Composite Hypotheses," Commun. Stat. Theory and Methods **39** (3), 460–471 (2010).
42. B. Yu. Lemeshko, S. B. Lemeshko, K. A. Akushkina, et al., "Inverse Gaussian Model and its Applications in Reliability and Survival Analysis" in *Mathematical and Statistical Models and Methods in Reliability. Applications to Medicine* (Birkhäuser, Boston, 2011), pp. 433–453.
43. B. Yu. Lemeshko and S. B. Lemeshko, "Models of Statistic Distributions of Nonparametric Goodness-of-Fit Tests in Composite Hypotheses Testing for Double Exponential Law Cases," Commun. Stat. Theory and Methods **40** (16), 2879–2892 (2011).
44. B. Yu. Lemeshko and S. B. Lemeshko, "Construction of Statistic Distribution Models for Nonparametric Goodness-of-Fit Tests in Testing Composite Hypotheses: The Computer Approach," Quality Technol. Quantitative Management **8** (4), 359–373 (2011).

45. B. Yu. Lemeshko, S. B. Lemeshko, and A. P. Rogozhnikov, "Real-Time Studying of Statistic Distributions of Non-Parametric Goodness-of-Fit Tests when Testing Complex Hypotheses," in *Proc. of the International Workshop "Applied Methods of Statistical Analysis. Simulations and Statistical Inference",* Novosibirsk, Russia, 20–22 Sept., 2011, 19–27.

46. *ISW — Software for Statistical Analysis of One-Dimensional Random Variables.* http://www.ami.nstu.ru/~headrd/ISW.htm.

47. B. Yu. Lemeshko, E. V. Chimitova, and T. A. Pleshkova, "Testing Simple and Composite Goodness-of-Fit Tests from Censored Samples," Nauch. Vestn. NGTU, No. 4, 13–28 (2010).

48. B. Yu. Lemeshko, E. V. Chimitova, and M. A. Vedernikova, "Modified Kolmogorov, Cramer–von Mises–Smirnov, and Anderson–Darling Goodness-of-Fit Tests for Randomly Censored Samples. Pt. 1," Nauch. Vestn. NGTU, No. 4, 12–19 (2012).

49. N. S. Galanova, B. Yu. Lemeshko, and E. V. Chimitova, "Using Nonparametric Goodness-of-Fit Tests to Validate Failure Time Models of Accelerated Tests," Avtometriya **48** (6), 53–68 (2012) [Optoelectron., Instrum. Data Process. **48** (6), 580–592 (2012)].