# A REVIEW OF THE PROPERTIES OF SOME TESTS FOR EXPONENTIALITY

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***Abstract*** **– A wide selection of statistical tests for testing exponentiality is considered. The distributions of tests statistics under true null hypothesis have been studied depending on sample sizes. Comparative analysis of the power of tests under different pairs of competing hypotheses has been conducted. Advantages and disadvantages of individual tests have been shown. Considered tests have been ranked by the test power. The conclusions are made on preference of one test or another under presence of some competing alternatives.**

***Index terms*** **– hypothesis testing, exponential distribution, power of test, order statistics.**

## I. INTRODUCTION

The exponential distribution law is one of common distribution in applied mathematics statistics. It is main distribution law used in reliability theory. Its analytical simplicity makes it attractive to engineers and researchers. However, you need to be sure, that behavior of observable random variable (for example, the moments of product failure (breakdown)) is consistent by desirable exponential distribution before using this model. Otherwise, the benefit from computation simplicity will be repeatedly reduced by losses from conclusion incorrectness caused by deviation of empirical distribution from exponential distribution law.

There are a lot of papers devoted to exponential law; authors of these papers propose different statistical tests for testing hypothesis of exponentiality. The abundance of tests is caused by frequent use of exponential distribution model in applications. However, the frequency of using is defined that usage of simple model leads to the solution of problem grounded only on analytical methods in most cases.

The presence of numerous tests put not simple problem of choosing for specialists, because available information in papers doesn't allows to give preference to certain test, while every specialist is interested not only in correctness of using of tests, but else in reliability of statistical inferences.

In this paper, considered tests have been studied by the method of statistical simulations. The number of experiments carried out for statistical modeling is usually assumed equal to $N = 1.66\times10^6$ in the study of the distributions of test statistics. One the one hand, such number of experiments allows tracing the qualitative picture of test statistic distributions in depend on various factors. In the other hand, this number of experiments provides acceptable accuracy of the power estimates and unknown probabilities. Computer analysis methods provide an opportunity to identify the advantages and disadvantages of a test, to assess the size of sample when the difference between distributions of test statistics under true tested hypothesis and the corresponding asymptotic (limiting) distributions of statistics is practically negligible. Also, these methods provide an opportunity to compare the relative powers of the different tests under various alternative hypotheses, and to identify the most preferable test.

## II. PROBLEM DEFINITION

Let $x_1, x_2, ..., x_n$ be sample of independent observations of nonnegative random variable $X$. Belonging of sample to exponential distribution law with density function $f(x) = \exp(-x)$ was considered as tested hypothesis $H_0$.

The set of tests constructed special for exponentiality testing can be used for testing hypothesis $H_0$ besides classical goodness-of-fit tests. It is quite difficult to divide the special test statistics into the groups due to multiplicity of its. It should be noted that elements of ordered samples $x_{(1)} \le x_{(2)} \le ... \le x_{(n)}$ are used in calculation for part of test statistics. In another cases sequence order of elements doesn't matter. Also, in some test statistics, we will use transformed values which use estimates of shift and scale or differences between elements of order samples.

The exponential distribution has constant failure rate. In view of this the distribution laws, belonging to three classes: with increasing, decreasing and non-monotonic failure rate [1] were considered as alternatives hypotheses.

The research was carried out for three alternative hypotheses:

$H_1$ : $LN(1)$ is lognormal distribution with density function $f(x) = \left(\theta x\sqrt{2\pi}\right)^{-1}\exp\left(-(\ln x)^2 / 2\theta^2\right)$ and scale parameter $\theta = 1$ as alternative hypothesis with non-monotonic failure rate;

$H_2$ : $W(0.7)$ is Weibull distribution with density function $f(x) = \theta x^{\theta-1}\exp\left(-x^{\theta}\right)$, and form parameter $\theta = 0.7$ as alternative hypothesis with decreasing failure rate;



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$H_3 : W(1.2)$ – is Weibull distribution with form parameter $\theta = 1.2$ as alternative hypothesis with increasing failure rate.

The distribution functions and density functions corresponding to tested and alternative hypotheses are presented on figures 1 and 2, respectively.

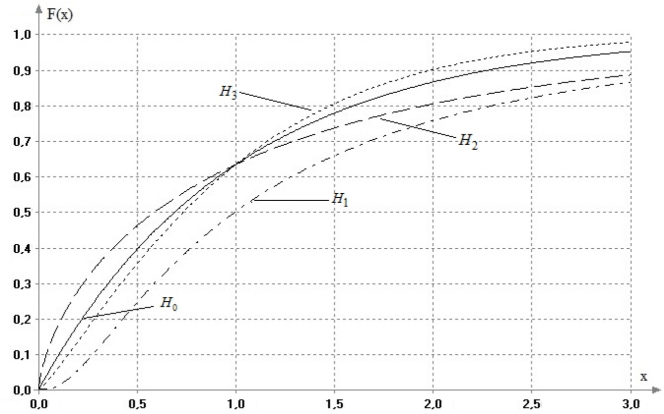


Fig. 1 – The distribution functions of hypotheses

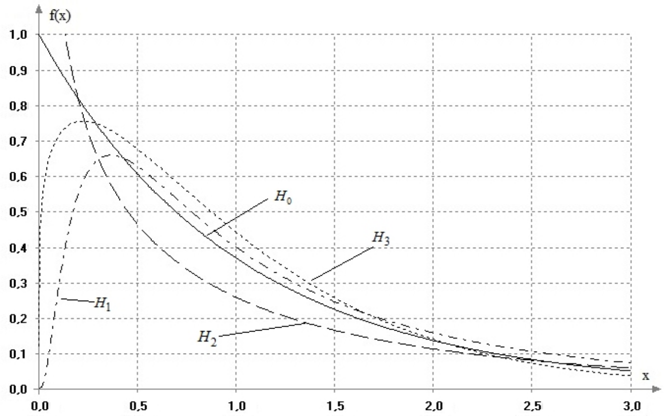


Fig. 2 – The density functions of hypotheses

## III. CONSIDERED TESTS

### A. Bartlett-Moran test

The Bartlett papers [2] are base for given test. The test statistic is:

$$B = \frac{12n^2}{7n+1}\left[\ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) + \frac{1}{n}\sum_{i=1}^{n}\ln x_i\right]. \qquad (1)$$

The distributions of the test statistic (1) of Bartlett-Moran test under $H_0$ and alternatives $H_1 - H_3$ for sample size $n = 50$ present on fig.3. You can see that this test is two-sided because distributions of test statistic under alternatives are offset in both directions from distribution of null hypothesis.

In [3] it is shown, that the distribution of test statistic is approximated by $\chi^2$ – distribution with $(n-1)$ degrees of freedom. The distributions of the test statistic of Bartlett-Moran test for $n = 100$ and some $\chi^2$ – distributions are shown in fig.4.

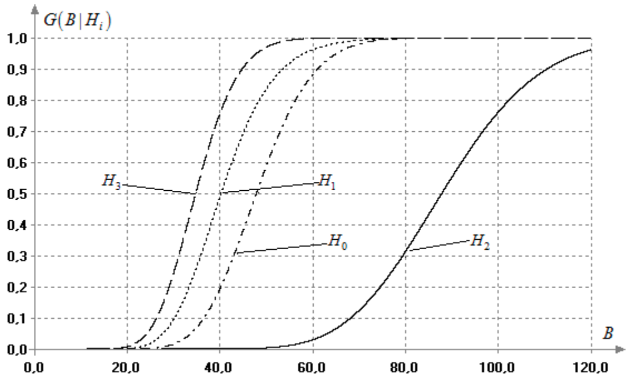


Fig. 3 – The distribution of test statistic (1) under $H_0 - H_3$ for $n = 50$

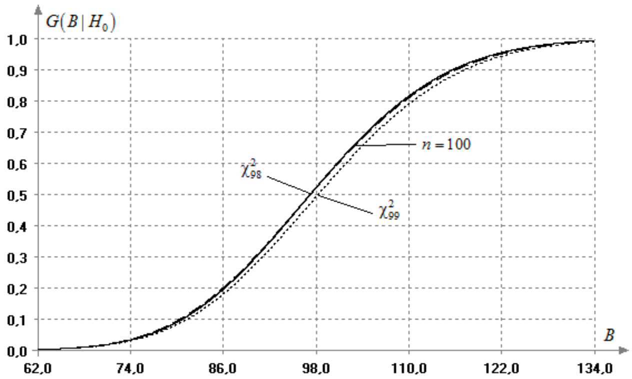


Fig. 4 – The distribution of test statistic (1) and $\chi^2$ – distributions

### B. Frosini Test

The test statistic is [4]:

$$B_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left|1 - \exp\left(-\frac{x_i}{\bar{x}}\right) - \frac{i-0,5}{n}\right|. \qquad (2)$$

The hypothesis under test is rejected for large values of the statistic. Critical values of the test statistic obtained by our simulation are presented in Table I. These values do not change for $n \to \infty$, it indicates the presence of the limit distribution. This test has likeness to Frosini test for uniformity [5]; however critical values and limit distribution of (2) differ from ones of uniformity test.

TABLE I
CRITICAL VALUES OF FROCINI TEST STATISTIC (2)

| $n$ | α | | | $n$ | α | | |
|---|---|---|---|---|---|---|---|
| | 0.9 | 0.95 | 0.99 | | 0.9 | 0.95 | 0.99 |
| 5 | 0.326 | 0.367 | 0.445 | 30 | 0.338 | 0.383 | 0.476 |
| 6 | 0.327 | 0.370 | 0.455 | 35 | 0.338 | 0.384 | 0.477 |
| 7 | 0.329 | 0.373 | 0.459 | 40 | 0.338 | 0.384 | 0.477 |
| 8 | 0.331 | 0.375 | 0.462 | 50 | 0.339 | 0.384 | 0.478 |
| 9 | 0.333 | 0.377 | 0.464 | 100 | 0.340 | 0.385 | 0.480 |
| 10 | 0.333 | 0.377 | 0.466 | 150 | 0.340 | 0.385 | 0.480 |
| 15 | 0.336 | 0.380 | 0.472 | 200 | 0.340 | 0.3855 | 0.480 |
| 20 | 0.337 | 0.3815 | 0.474 | 300 | 0.340 | 0.386 | 0.480 |
| 25 | 0.338 | 0.383 | 0.4755 | 500 | 0.340 | 0.3855 | 0.481 |

### C. Pietra test

This test named different in some papers. In [6, 7, 8] test called Pietra test, but in [3] it called as Sherman exponentiality test. The test statistic is

$$P_n = \frac{1}{2n}\frac{\sum_{i=1}^{n}\left|x_i - \bar{x}\right|}{\bar{x}}. \quad (3)$$

The critical values of test statistics (3) equal to ones of Sherman uniformity test [2, 10] in the opinion of some works [2]. It is true for large sample sizes, however for small sample sizes it is necessary to replace $n$ by $n+1$ in (3). The distribution of Pietra test statistic (3) and distribution of Sherman uniformity test denoted by $\omega_n$ are present on fig.5. The critical values of test statistic (3) are shown in Table II.

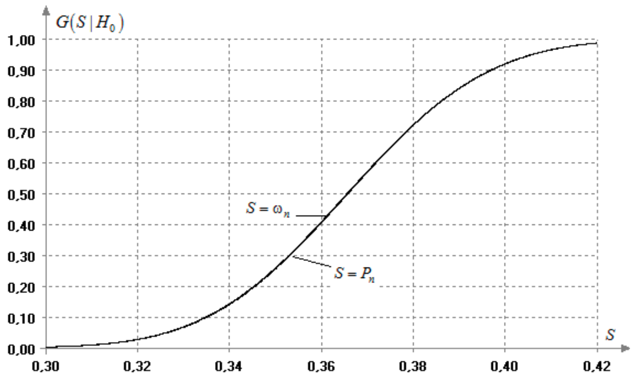


Fig. 5 – The distributions of Pietra test statistic (3) and Sherman uniformity test statistic

Pietra test is two-sided. The distributions of the test statistic (3) of Pietra test under $H_0$ and alternatives $H_1 - H_3$ for sample size $n=50$ present on fig.6.

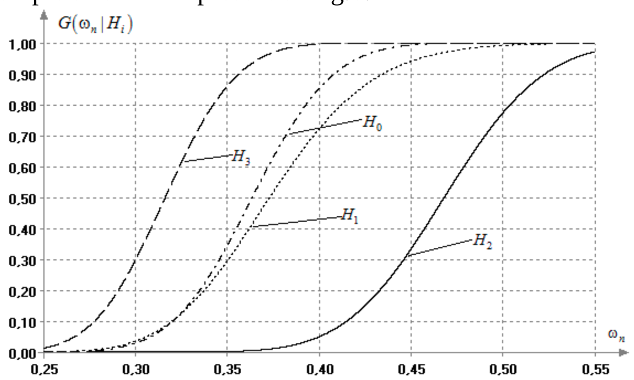


Fig. 6 – The distribution of test statistic (3) under $H_0 - H_3$ for $n=50$

There are two normal approximations for Sherman uniformity test, which can be used to Pietra test because distributions of these tests quite close.

The test statistic of first is:

$$\omega_n^* = \frac{P_n - \mathrm{E}[P_n]}{\sqrt{D[P_n]}}; \quad (4)$$

where

$$E[P_n] = \left(\frac{n}{n+1}\right)^{n+1} \approx \frac{1}{e};$$

$$D[P_n] = \frac{2n^2 + n(n-1)^{n+2}}{(n+2)(n+1)^{n+2}} - \left(\frac{n}{n+1}\right)^{2n+2}.$$

You can see closeness of distribution of test statistic (4) for $n=100$ to standard normal distribution law in fig.7.

TABLE II
CRITICAL VALUES OF PIETRA TEST STATISTIC (3)

| $n$ | α | | | | | |
|---|---|---|---|---|---|---|
| | 0.9 | | 0.95 | | 0.99 | |
| | $P_{n,1}$ | $P_{n,2}$ | $P_{n,1}$ | $P_{n,2}$ | $P_{n,1}$ | $P_{n,2}$ |
| 5 | 0.163 | 0.509 | 0.138 | 0.544 | 0.092 | 0.622 |
| 10 | 0.228 | 0.478 | 0.208 | 0.504 | 0.170 | 0.555 |
| 15 | 0.256 | 0.460 | 0.238 | 0.481 | 0.205 | 0.523 |
| 20 | 0.272 | 0.449 | 0.256 | 0.468 | 0.227 | 0.503 |
| 30 | 0.290 | 0.436 | 0.277 | 0.450 | 0.253 | 0.479 |
| 40 | 0.301 | 0.427 | 0.290 | 0.440 | 0.268 | 0.465 |
| 50 | 0.308 | 0.421 | 0.298 | 0.433 | 0.278 | 0.455 |
| 100 | 0.326 | 0.406 | 0.319 | 0.414 | 0.305 | 0.430 |
| 150 | 0.334 | 0.400 | 0.328 | 0.406 | 0.316 | 0.419 |
| 200 | 0.339 | 0.395 | 0.334 | 0.401 | 0.323 | 0.412 |
| 300 | 0.344 | 0.390 | 0.340 | 0.395 | 0.332 | 0.404 |

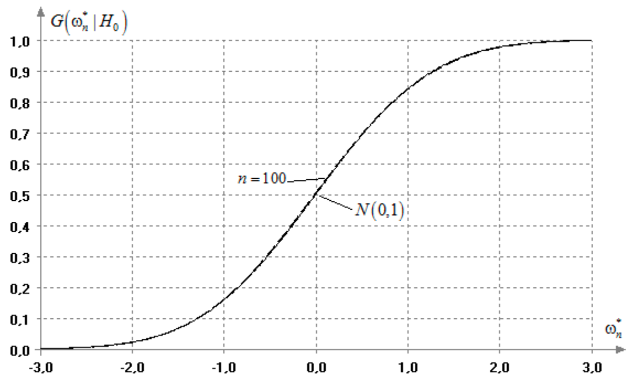


Fig. 7 – The distribution of test statistic (4) and normal distribution

Another normal approximation [9] is described by formulas:

$$\tilde{\omega}_n = U - \frac{0,0995}{\sqrt{n}}\left(U^2 - 1\right); \quad (5)$$

where

$$U = \frac{P_n - 0,3679\left(1 - \frac{1}{2n}\right)}{\frac{0,2431}{\sqrt{n}}\left(1 - \frac{0,605}{n}\right)}.$$

The distribution of this modified test statistic (5) faster converges to the standard normal distribution than the distribution of test statistic (4). The convergence of distribution of test statistic (5) to the standard normal distribution law is shown on fig.8.

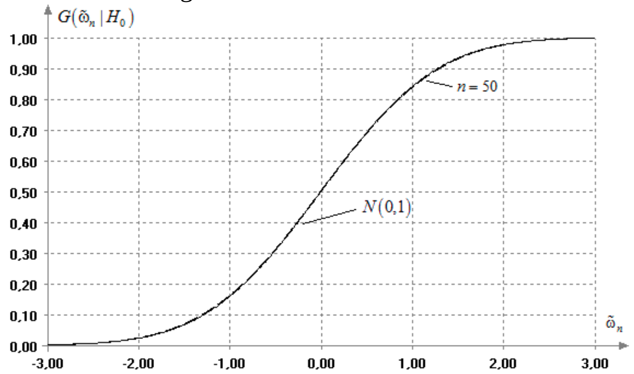


Fig. 8 – The distribution of test statistic (5) and normal distribution

D. Kochar test

The statistic of test is [10]:

$$T_n = \sqrt{\frac{108}{n}} \frac{\sum_{i=1}^{n} J\left(\frac{i}{n+1}\right)}{\sum_{i=1}^{n} x_i}; \qquad (6)$$

where

$$J\left(\frac{i}{n+1}\right) = 2\left(\frac{n+1-i}{n+1}\right)\left[1-\ln\left(\frac{n+1-i}{n+1}\right)\right]-1.$$

The research shows that Kochar test is two-sided. Figure 9 containing the distributions of the test statistic (6) of Kochar test under $H_0$ and alternatives for sample size $n = 50$ confirms that statement. The critical values of Kochar test are demonstrated in Table III.

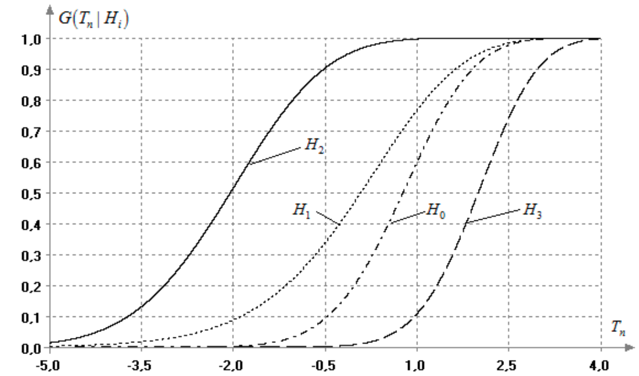


Fig. 9 – The distribution of test statistic (6) under $H_0 - H_3$ for $n = 50$

E. Lawless test

The test [11] constructed for exponentiality testing against gamma-distribution alternatives based on test statistic:

$$W = \frac{\tilde{x}}{\bar{x}}, \qquad (7)$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\tilde{x} = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$ — arithmetic mean and geometric mean of sample $x_1, x_2, ..., x_n$, respectively.

This test is two-sided; the hypothesis of exponentiality is rejected for both small and large values of test statistic. The critical values of test statistic (7) are presented in Table IV.

It is worth nothing that test statistic (7) is generalization of test statistic (1) of Bartlett-Moran test. The powers of both tests are identical for all considered situations in own research.

TABLE III
CRITICAL VALUES OF KOCHAR TEST STATISTIC (6)

| $n$ | α | | | | | |
|---|---|---|---|---|---|---|
| | 0.9 | | 0.95 | | 0.99 | |
| | $P_{n,1}$ | $P_{n,2}$ | $P_{n,1}$ | $P_{n,2}$ | $P_{n,1}$ | $P_{n,2}$ |
| 5 | 0.750 | 2.486 | 0.568 | 2.606 | 0.251 | 2.814 |
| 10 | 0.175 | 2.490 | -0.078 | 2.664 | -0.574 | 2.976 |
| 15 | -0.129 | 2.443 | -0.413 | 2.644 | -0.971 | 3.013 |
| 20 | -0.326 | 2.396 | -0.622 | 2.615 | -1.208 | 3.023 |
| 25 | -0.464 | 2.356 | -0.767 | 2.588 | -1.375 | 3.018 |
| 30 | -0.564 | 2.322 | -0.876 | 2.559 | -1.498 | 3.009 |
| 40 | -0.714 | 2.266 | -1.031 | 2.517 | -1.667 | 2.985 |
| 50 | -0.814 | 2.218 | -1.134 | 2.479 | -1.771 | 2.972 |
| 100 | -1.066 | 2.087 | -1.393 | 2.365 | -2.038 | 2.898 |
| 150 | -1.178 | 2.020 | -1.504 | 2.308 | -2.144 | 2.855 |
| 200 | -1.241 | 1.978 | -1.565 | 2.270 | -2.209 | 2.828 |
| 300 | -1.317 | 1.922 | -1.640 | 2.217 | -2.284 | 2.795 |

TABLE IV
CRITICAL VALUES OF LAWLESS TEST STATISTIC (7)

| $n$ | α | | | | | |
|---|---|---|---|---|---|---|
| | 0.9 | | 0.95 | | 0.99 | |
| | $W_1$ | $W_2$ | $W_1$ | $W_2$ | $W_1$ | $W_2$ |
| 5 | 0.330 | 0.917 | 0.274 | 0.943 | 0.181 | 0.975 |
| 10 | 0.376 | 0.821 | 0.334 | 0.851 | 0.259 | 0.902 |
| 15 | 0.402 | 0.772 | 0.367 | 0.801 | 0.303 | 0.852 |
| 20 | 0.419 | 0.743 | 0.389 | 0.770 | 0.331 | 0.818 |
| 25 | 0.432 | 0.723 | 0.404 | 0.748 | 0.352 | 0.793 |
| 30 | 0.442 | 0.708 | 0.416 | 0.731 | 0.368 | 0.774 |
| 40 | 0.456 | 0.688 | 0.434 | 0.708 | 0.391 | 0.746 |
| 50 | 0.466 | 0.674 | 0.446 | 0.692 | 0.408 | 0.728 |
| 100 | 0.492 | 0.6395 | 0.478 | 0.653 | 0.450 | 0.680 |
| 150 | 0.504 | 0.625 | 0.492 | 0.636 | 0.470 | 0.658 |
| 200 | 0.511 | 0.616 | 0.501 | 0.626 | 0.482 | 0.645 |
| 300 | 0.520 | 0.606 | 0.512 | 0.614 | 0.496 | 0.629 |

F. Greenwood test

The Greenwood exponentiality test has the following test statistic:

$$G=\frac{\mathrm{E}\left[X^2\right]}{\left(\mathrm{E}\left[X\right]\right)^2}=\frac{\frac{1}{n}\sum_{i=1}^{n}x_i^2}{\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2}=n\frac{\sum_{i=1}^{n}x_i^2}{\left(\sum_{i=1}^{n}x_i\right)^2}. \quad (8)$$

This test is two-sided; the hypothesis of exponentiality is rejected for both small and large values of test statistic. The critical values of test statistic (8) are shown in Table V. As said in [2] the critical values of Greenwood test statistic (8) for sample sizes $n+1$ coincided with ones of uniformity Greenwood test statistic for sample sizes $n$. The distributions of both test statistics for their true hypotheses and $n=50$ are demonstrated on fig.10. The description of Greenwood uniformity test is presented in [5].

These distributions quite close. This proves that previous statement is true and difference between sample sizes is negligible for large values of $n$.

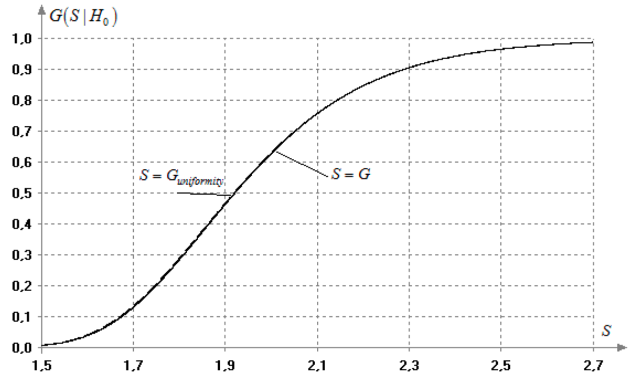


Fig. 10 – The distributions of exponentiality and uniformity of Greenwood test statistics

TABLE V
CRITICAL VALUES OF GREENWOOD TEST STATISTIC (8)

| $n$ | α | | | | | |
|---|---|---|---|---|---|---|
| | 0,9 | | 0,95 | | 0,99 | |
| | $G_1$ | $G_2$ | $G_1$ | $G_2$ | $G_1$ | $G_2$ |
| 5 | 1.154 | 2.545 | 1.109 | 2.839 | 1.049 | 3.442 |
| 10 | 1.311 | 2.646 | 1.261 | 2.932 | 1.179 | 3.633 |
| 15 | 1.398 | 2.624 | 1.347 | 2.882 | 1.261 | 3.521 |
| 20 | 1.456 | 2.590 | 1.405 | 2.818 | 1.319 | 3.395 |
| 30 | 1.531 | 2.525 | 1.483 | 2.714 | 1.400 | 3.192 |
| 40 | 1.581 | 2.476 | 1.534 | 2.641 | 1.455 | 3.049 |
| 50 | 1.617 | 2.438 | 1.572 | 2.583 | 1.494 | 2.938 |
| 100 | 1.713 | 2.327 | 1.675 | 2.424 | 1.608 | 2.651 |
| 150 | 1.760 | 2.271 | 1.726 | 2.347 | 1.666 | 2.521 |
| 200 | 1.789 | 2.236 | 1.758 | 2.300 | 1.702 | 2.443 |
| 300 | 1.824 | 2.194 | 1.798 | 2.243 | 1.748 | 2.352 |

F. Epstein test

The test statistic is [6, 7, 8]:

$$EPS_n=\frac{2n\left(\ln\left(\frac{1}{n}\sum_{i=1}^{n}D_i\right)-\frac{1}{n}\sum_{i=1}^{n}\ln\left(D_i\right)\right)}{1+\left(n+1\right)/\left(6n\right)}, \quad (9)$$

where

$$D_i=\left(n-i+1\right)\left(x_{(i)}-x_{(i-1)}\right),\ x_{(0)}=0.$$

This test statistic (9) is test statistic (1) of Bartlett-Moran under replace $x_i$ on $D_i$. As you can see $D_i$ estimates difference between neighbors in ordered statistics $x_{(1)}\le x_{(2)}\le\ldots\le x_{(n)}$.

This test is two-sided. The distributions of the test statistic (9) under alternatives $H_1-H_3$ differ in both directions from the distribution of test statistic under null hypothesis (see fig.11).

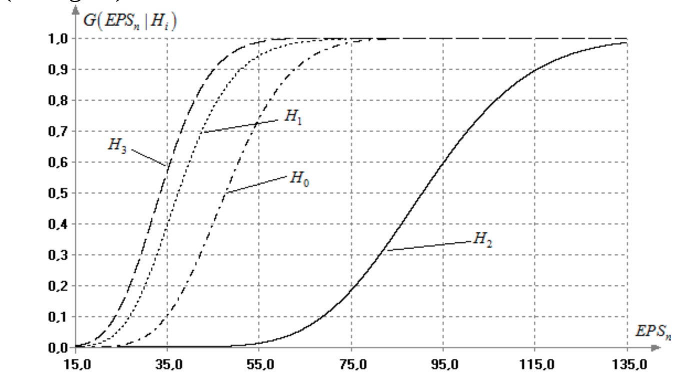


Fig. 11 – The distributions of test statistic (9) under $H_0-H_3$ for $n=50$

In [8] said that distribution of $G\left(EPS_n\,|\,H_0\right)$ is chi-squared distribution with $(n-1)$ degree of freedom. However there is some distinction between the distributions, the example for $n=100$ are shown in fig.12.

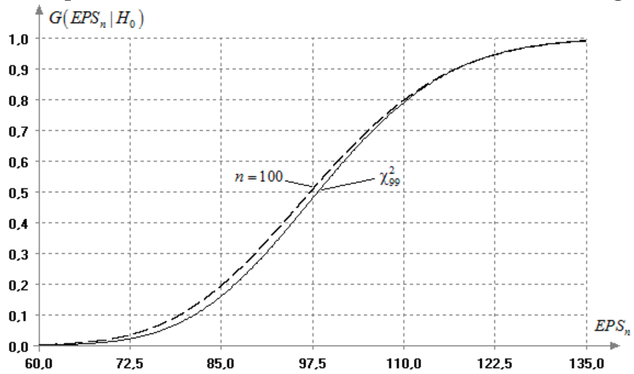


Fig. 12 – The distribution of test statistic (9) and $\chi^2$ – distribution

G. Moran test

The test statistic is [8]:

$$T_n^{+}=\gamma+\frac{1}{n}\sum_{i=1}^{n}\ln\frac{x_i}{\bar{x}}; \quad (10)$$

where $\gamma=0.577215$ is Euler–Mascheroni constant.

The normalized test statistic for this test proposed in [11] is:

$$T_n^{+,*} = \frac{T_n^+ - \mathrm{E}\left[T_n^+\right]}{\sqrt{D\left[T_n^+\right]}}; \qquad (11)$$

where $\mathrm{E}\left[T_n^+\right] = 0,\ D\left[T_n^+\right] = \frac{1}{n}\left(\frac{\pi^2}{6} - 1\right)^2$.

The convergence of (11) to the standard normal distribution law is quite “slow” and the distribution of test statistic is described by normal distribution only for large sample sizes. You can see that convergence on fig.13, where the distributions of test statistic for $n = 100$, $n = 500$ and standard normal distribution law are presented.

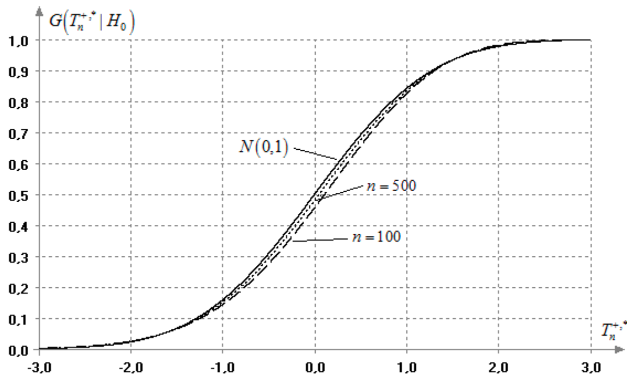


Fig. 13 – The distributions of test statistic (11) and normal distribution law

### H. Hegazy-Green test

The Hegazy-Green exponentiality test has the following test statistics:

$$T_1 = \frac{1}{n}\sum_{i=1}^{n}\left| x_{(i)} + \ln\left(1 - \frac{i}{n+1}\right)\right|; \qquad (12)$$

$$T_2 = \frac{1}{n}\sum_{i=1}^{n}\left( x_{(i)} + \ln\left(1 - \frac{i}{n+1}\right)\right)^2. \qquad (13)$$

This test is right-sided like ones for normality and uniformity [5] testing.

The critical values of test statistics obtained by modeling are shown in Table VI.

This is worth nothing that considered test of Hegazy-Green has serious weakness under testing against $H_3$ type of alternative hypotheses in power analysis.

TABLE VI
CRITICAL VALUES OF HEGAZY-GREEN TEST STATISTICS (11) AND (12)

| $n$ | $T_1$ | | | $T_2$ | | |
|---|---|---|---|---|---|---|
| | 0.9 | 0.95 | 0.99 | 0.9 | 0.95 | 0.99 |
| 10 | 0.583 | 0.718 | 1.013 | 0.800 | 1.257 | 2.630 |
| 20 | 0.412 | 0.493 | 0.678 | 0.470 | 0.713 | 1.428 |
| 30 | 0.337 | 0.398 | 0.538 | 0.338 | 0.505 | 0.985 |
| 40 | 0.292 | 0.342 | 0.457 | 0.267 | 0.393 | 0.755 |
| 50 | 0.261 | 0.304 | 0.403 | 0.222 | 0.014 | 0.614 |
| 100 | 0.184 | 0.213 | 0.277 | 0.123 | 0.175 | 0.323 |
| 150 | 0.150 | 0.173 | 0.224 | 0.087 | 0.122 | 0.221 |
| 200 | 0.130 | 0.150 | 0.193 | 0.068 | 0.094 | 0.169 |
| 300 | 0.107 | 0.122 | 0.156 | 0.047 | 0.065 | 0.115 |

TABLE VII
THE EXPONENTIALITY TESTS RANKED BY POWER UNDER $n = 50$ AND $\alpha = 0.05$

| № | hypothesis $H_1$ | $1-\beta$ | hypothesis $H_2$ | $1-\beta$ | hypothesis $H_3$ | $1-\beta$ |
|---|---|---|---|---|---|---|
| 1 | Hegazy-Green test 1 | 0.925 | Bartlett-Moran test | 0.890 | Bartlett-Moran test | 0.315 |
| 2 | Hegazy-Green test 2 | 0.882 | Lawless test | 0.890 | Lawless test | 0.315 |
| 3 | Kimber-Michael test | 0.516 | Moran test | 0.890 | Moran test | 0.315 |
| 4 | Correlation test 2 | 0.377 | Epstein test | 0.863 | Epps-Pulley test | 0.304 |
| 5 | Klimko-Antle-Rademaker-Rockette test | 0.359 | Epps-Pulley test | 0.831 | Frosini test | 0.291 |
| 6 | Shapiro-Wilk test 1 | 0.359 | Hollander-Proshan test | 0.818 | Epstein test | 0.288 |
| 7 | Correlation test 1 | 0.344 | Pietra/Sherman test | 0.804 | Hollander-Proshan test | 0.280 |
| 8 | Frosini test | 0.310 | Frosini test | 0.804 | Kimber-Michael test | 0.279 |
| 9 | Shapiro-Wilk test 2 | 0.290 | Kochar test | 0.772 | Pietra/Sherman test | 0.277 |
| 10 | Greenwood test | 0.290 | Hegazy-Green test 2 | 0.772 | Kochar test | 0.268 |
| 11 | Max interval test | 0.254 | Hegazy-Green test 1 | 0.761 | Greenwood test | 0.267 |
| 12 | Kochar test | 0.218 | Kimber-Michael test | 0.706 | Shapiro-Wilk test 2 | 0.266 |
| 13 | Epps-Pulley test | 0.171 | Shapiro-Wilk test 2 | 0.657 | Klimko-Antle-Rademaker-Rockette test | 0.223 |
| 14 | Epstein test | 0.166 | Greenwood test | 0.657 | Shapiro-Wilk test 1 | 0.223 |
| 15 | Bartlett-Moran test | 0.143 | Klimko-Antle-Rademaker-Rockette test | 0.621 | Max interval test | 0.125 |
| 16 | Lawless test | 0.143 | Shapiro-Wilk test 1 | 0.621 | Correlation test 1 | 0.053 |
| 17 | Moran test | 0.143 | Max interval test | 0.347 | Correlation test 2 | 0.016 |
| 18 | Pietra/Sherman test | 0.140 | Correlation test 2 | 0.311 | Hegazy-Green test 1 | 0.009 |
| 19 | Hollander-Proshan test | 0.109 | Correlation test 1 | 0.276 | Hegazy-Green test 2 | 0.001 |

TABLE VIII
THE EXPONENTIALITY TESTS RANKED BY POWER UNDER $n = 100$ AND $\alpha = 0.05$

| № | hypothesis $H_1$ | $1-\beta$ | hypothesis $H_2$ | $1-\beta$ | hypothesis $H_3$ | $1-\beta$ |
|---|---|---|---|---|---|---|
| 1 | Hegazy-Green test 1 | 0.996 | Bartlett-Moran test | 0.993 | Bartlett-Moran test | 0.582 |
| 2 | Hegazy-Green test 2 | 0.985 | Lawless test | 0.993 | Lawless test | 0.582 |
| 3 | Kimber-Michael test | 0.876 | Moran test | 0.993 | Moran test | 0.582 |
| 4 | Frosini test | 0.585 | Epstein test | 0.990 | Epps-Pulley test | 0.570 |
| 5 | Correlation test 2 | 0.548 | Epps-Pulley test | 0.982 | Epstein test | 0.552 |
| 6 | Klimko-Antle-Rademaker-Rockette test | 0.541 | Hollander-Proshan test | 0.980 | Frosini test | 0.533 |
| 7 | Shapiro-Wilk test 1 | 0.541 | Frosini test | 0.977 | Hollander-Proshan test | 0.530 |
| 8 | Correlation test 1 | 0.525 | Pietra/Sherman test | 0.974 | Pietra/Sherman test | 0.520 |
| 9 | Shapiro-Wilk test 2 | 0.440 | Kochar test | 0.963 | Kochar test | 0.510 |
| 10 | Greenwood test | 0.440 | Hegazy-Green test 1 | 0.948 | Shapiro-Wilk test 2 | 0.489 |
| 11 | Max interval test | 0.366 | Kimber-Michael test | 0.947 | Greenwood test | 0.489 |
| 12 | Kochar test | 0.320 | Hegazy-Green test 2 | 0.938 | Kimber-Michael test | 0.472 |
| 13 | Epstein test | 0.258 | Shapiro-Wilk test 2 | 0.906 | Klimko-Antle-Rademaker-Rockette test | 0.438 |
| 14 | Bartlett-Moran test | 0.218 | Greenwood test | 0.906 | Shapiro-Wilk test 1 | 0.438 |
| 15 | Lawless test | 0.218 | Shapiro-Wilk test 1 | 0.893 | Max interval test | 0.185 |
| 16 | Moran test | 0.218 | Klimko-Antle-Rademaker-Rockette test | 0.893 | Correlation test 1 | 0.060 |
| 17 | Epps-Pulley test | 0.214 | Max interval test | 0.462 | Hegazy-Green test 1 | 0.033 |
| 18 | Hollander-Proshan test | 0.202 | Correlation test 2 | 0.442 | Correlation test 2 | 0.019 |
| 19 | Pietra/Sherman test | 0.167 | Correlation test 1 | 0.422 | Hegazy-Green test 2 | 0.006 |

## IV. POWER ANALYSIS

The Tables VII and VIII contain considered tests ordered by decreasing of power (quantity $1-\beta$) under alternatives $H_1$, $H_2$ and $H_3$ (under sample sizes $n=50$, $n=100$ and significance level $\alpha = 0.05$). The description of another exponentiality tests demonstrated in paper [13]. The dark mark means that the test is biased, in other words that quantity $\alpha$ larger than $1-\beta$.

The best results are shown by Hegazy-Green test under hypothesis $H_1$, the power of this test are larger than powers of other tests considered. However tests of Hegazy-Green have large biasness under hypothesis $H_3$.The Kimball-Michael test has less power than Hegazy-Green tests under first alternative, but it has good powers under another hypotheses. Frosini test lose to previous tests but still show good power, especially under $n=100$. Also Frosini test has larger powers than Hegazy-Green tests and Kimber-Michael test under hypotheses $H_2 - H_3$.

It should be noted that worst results under hypothesis H1 are shown by tests, limit distribution of which is normal distribution law (Pietra test, Hollander- Proshan test, etc).

The Bartlett-Moran test, Lawless test and Moran test, which powers are identical, show the highest powers under alternative hypotheses $H_2$ and $H_3$. However, these tests demonstrate average power under hypothesis $H_1$. The Epps-Pulley test, Frosini test, Hollander-Proshan test and Epstein test demonstrate consistently good ability to distinguish those alternative hypotheses from exponential distribution as well.

The low power are presented here by correlation tests (especially under hypothesis $H_3$) and max interval test.

As mentioned before Hegazy-Green test shows worst power under hypothesis with increasing failure rate. This weakness is demonstrated in work [6].

## V. CONCLUSIONS

The Frosini test and Kimber-Michael test are recommended for usage from considered tests. It is worth noting that Frosini test has limit distribution as well.

In spite of advantages, the application of Barltett-Moran test (and tests equal to it) and Pietra test is quite risky because its powers under hypotheses with non-monotonic failure rate isn't good enough. A similar situation arises for application of Hegazy-Green test under hypotheses increasing failure rate.

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