Chapter 31
Application of Nonparametric Goodness-of-Fit Tests for Composite Hypotheses in Case of Unknown Distributions of Statistics

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31.1 Introduction

Classical nonparametric tests were constructed for testing simple hypotheses: $H_0 : F(x) = F(x, \theta)$, where θ is known scalar or vector parameter of the distribution function $F(x, \theta)$. When testing simple hypotheses nonparametric criteria are distribution free, i.e. the distribution $G(S|H_0)$, where S is the test statistic, does not depend on the $F(x, \theta)$ when the hypothesis H_0 is true.

When testing composite hypotheses $H_0 : F(x) \in \{F(x, \theta), \theta \in \Theta\}$, where the estimate $\hat{\theta}$ of a scalar or vector parameter of the distribution $F(x, \theta)$ is calculated from the same sample, nonparametric tests lose the distribution freedom. Conditional distributions $G(S|H_0)$ of tests statistics for composite hypotheses depend on a number of factors: the type of the distribution $F(x, \theta)$, corresponding to the true hypothesis H_0; the type of the estimated parameter and the number of estimated parameters and, in some cases, the value of the parameter; the method of the parameter estimation.

31.2 Nonparametric Goodness-of-Fit Criteria for Testing Simple Hypotheses

In **Kolmogorov test** statistic the distance between the empirical and theoretical distribution is determined by

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V.B. Melas et al. (eds.), *Topics in Statistical Simulation*, Springer Proceedings in Mathematics & Statistics 114, DOI 10.1007/978-1-4939-2104-1_31

$$D_n = \sup_{|x|<\infty} |F_n(x) - F(x,\theta)|,$$

where $F_n(x)$ is the empirical distribution function, n is the sample size. When $n \to \infty$, distribution of statistic $\sqrt{n}D_n$ for true hypothesis under test uniformly converges to the Kolmogorov distribution [15]

$$K(S) = \sum_{k=-\infty}^{\infty} (-1)^k e^{-2k^2 s^2}.$$

While testing hypothesis using the Kolmogorov test it is advisable to use the statistic with Bolshev correction [4] given by [5]:

$$S_K = \frac{6nD_n + 1}{6\sqrt{n}}, \tag{31.1}$$

where $D_n = \max(D_n^+, D_n^-)$,

$$D_n^+ = \max_{1 \leqslant i \leqslant n} \left\{ \frac{i}{n} - F(x_i,\theta) \right\}, D_n^- = \max_{1 \leqslant i \leqslant n} \left\{ F(x_i,\theta) - \frac{i-1}{n} \right\},$$

n is the sample size, $x_1, x_2, \ldots, x_n$ are the sample values in an increasing order. When a simple hypothesis H_0 under test is true, the statistic (31.1) converges to the Kolmogorov distribution significantly faster than statistic $\sqrt{n}D_n$.

The statistic of **Cramer–von Mises–Smirnov test** has the following form [3]:

$$S_\omega = \frac{1}{12n} + \sum_{i=1}^{n} \left\{ F(x_i,\theta) - \frac{2i-1}{2n} \right\}^2, \tag{31.2}$$

and **Anderson–Darling test** statistic [2,3] is

$$S_\Omega = -n - 2\sum_{i=1}^{n} \left\{ \frac{2i-1}{2n} \ln F(x_i,\theta) + \left(1 - \frac{2i-1}{2n}\right) \ln(1 - F(x_i,\theta)) \right\}. \tag{31.3}$$

When testing simple hypotheses, statistic (31.2) has the following distribution $a1(s)$ and the statistic (31.3) has the distribution $a2(s)$ [5].

The **Kuiper test** [16] is based on the statistic $V_n = D_n^+ + D_n^-$. The limit distribution of statistic $\sqrt{n}V_n$ while testing simple hypothesis is the following distribution function [36]:

$$G(s|H_0) = 1 - \sum_{m=1}^{\infty} 2(4m^2s^2 - 1)e^{-2m^2s^2}.$$

The following modification of the statistic converges faster to the limit distribution [38]:

$$V = V_n \left(\sqrt{n} + 0.155 + \frac{0.24}{\sqrt{n}} \right),$$

or the modification that we have chosen:

$$V_n^{\mathrm{mod}} = \sqrt{n}(D_n^+ + D_n^-) + \frac{1}{3\sqrt{n}}. \tag{31.4}$$

Dependence of the distribution of statistic (31.4) on the sample size is practically negligible when $n \geqslant 30$.

As a model of limit distribution we can use the beta distribution of the third kind with the density

$$f(s) = \frac{\theta_2^{\theta_0}}{\theta_3 B(\theta_0, \theta_1)} \frac{\left(\frac{s-\theta_4}{\theta_3}\right)^{\theta_0 - 1} \left(1 - \frac{s-\theta_4}{\theta_3}\right)^{\theta_1 - 1}}{\left[1 + (\theta_2 - 1)\frac{s-\theta_4}{\theta_3}\right]^{\theta_0 + \theta_1}},$$

and the vector of parameters $\boldsymbol{\theta} = (7.8624, 7.6629, 2.6927, 0.495)^T$, obtained by the simulation of the distribution of the statistic (31.4).

Watson test [41,42] is used in the following form

$$U_n^2 = \sum_{i=1}^{n} \left(F(x_i, \theta) - \frac{i - \frac{1}{2}}{n} \right)^2 - n \left(\frac{1}{n} \sum_{i=1}^{n} F(x_i, \theta) - \frac{1}{2} \right) + \frac{1}{12n}. \tag{31.5}$$

The limit distribution of the statistic (31.5) while testing simple hypotheses is given by [41,42]:

$$G(s|H_0) = 1 - 2\sum_{m=1}^{\infty} (-1)^{m-1} e^{-2m^2\pi^2 s}.$$

The good model for the limit distribution of the statistic (31.5) is the inverse Gaussian distribution with the density

$$f(s) = \frac{1}{\theta_2} \left(\frac{\theta_0}{2\pi \left(\frac{s-\theta_3}{\theta_2}\right)^2} \right)^{1/2} \exp\left(-\frac{\theta_0 \left(\left(\frac{s-\theta_3}{\theta_2}\right) - \theta_1 \right)}{2\theta_1^2 \left(\frac{s-\theta_3}{\theta_2}\right)} \right)$$

and the vector of parameters $\boldsymbol{\theta} = (0.2044, 0.08344, 1.0, 0.0)^T$, obtained by the simulation of the empirical distribution of the statistic (31.5). This distribution as well as the limit one could be used in testing simple hypotheses with Watson test to calculate the achieved significance level.

Zhang tests were proposed in papers [43–45]. The statistics of these criteria are:

$$Z_K = \max_{1 \leqslant i \leqslant n} \left(\left(i - \frac{1}{2} \right) \log \left\{ \frac{i - \frac{1}{2}}{n F(x_i, \theta)} \right\} + \left(n - i + \frac{1}{2} \right) \log \left[\frac{n - 1 + \frac{1}{2}}{n\{1 - F(x_i, \theta)\}} \right] \right), \tag{31.6}$$

$$Z_A = -\sum_{i=1} n \left[\frac{\log\{F(x_i, \theta)\}}{n - i + \frac{1}{2}} + \frac{\log\{1 - F(x_i, \theta)\}}{i - \frac{1}{2}} \right], \tag{31.7}$$

$$Z_C = \sum_{i=1} n \left[\log \left\{ \frac{[F(x_i, \theta)]^{-1} - 1}{\left(n - \frac{1}{2}\right) / \left(i - \frac{3}{4}\right) - 1} \right\} \right]^2. \tag{31.8}$$

The author gives the percentage points for statistics distributions for the case of testing simple hypotheses. The strong dependence of statistics distributions on the sample size n prevents one from wide use of the criteria with the statistics (31.6)–(31.8). For example, Fig. 31.1 shows a dependence of the distribution of the statistics (31.7) on the sample size while testing simple hypotheses.

Of course, this dependence on the sample size n remains for the case of testing composite hypotheses.

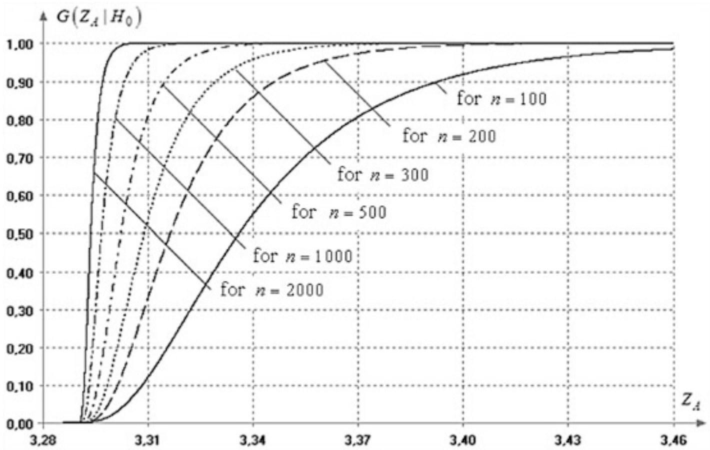

Fig. 31.1 The distribution $G_n(Z_A | H_0)$ of statistic (31.7) depending on the sample size n for testing simple hypothesis

31.3 Comparative Analysis of the Tests Power

In papers [25–27] the power of Kolmogorov (K), Cramer–von Mises–Smirnov (KMS), Anderson–Darling (AD) tests, and also χ^2 criteria was analyzed and compared for testing simple and composite hypotheses for a number of different pairs of competing distributions. In the case of testing simple hypotheses and using asymptotically optimal grouping [17] in χ^2 criterion, this test has the advantage in power compared with nonparametric tests [25, 26]. When testing composite hypotheses, the power of nonparametric tests increases significantly, and they become more powerful.

In order to be able to compare the power of Kuiper (V_n), Watson (U_n^2), and Zhang tests (Z_K, Z_A, Z_C) with the power of other goodness-of-fit tests, the power of these criteria was calculated for the same pairs of competing distributions in the paper [19] alike papers [25–27].

The first pair is the normal and logistics distribution: for the hypothesis H_0—the normal distribution with the density:

$$f(x) = \frac{1}{\theta_0\sqrt{2\pi}} \exp\left\{-\frac{(x-\theta_1)^2}{2\theta_0^2}\right\},$$

and for competing hypothesis H_1—the logistic distribution with the density:

$$f(x) = \frac{\pi}{\theta_1\sqrt{3}} \exp\left\{-\frac{\pi(x-\theta_0)}{\theta_1\sqrt{3}}\right\} \bigg/ \left[1+\exp\left\{-\frac{\pi(x-\theta_0)}{\theta_1\sqrt{3}}\right\}\right]^2,$$

and parameters $\theta_0 = 1$, $\theta_1 = 1$. For the simple hypothesis H_0 parameters of the normal distribution have the same values. These two distributions are close and difficult to distinguish with goodness-of-fit tests.

The second pair was the following: H_0—Weibull distribution with the density

$$f(x) = \frac{\theta_0(x-\theta_2)^{\theta_0-1}}{\theta_1^{\theta_0}} \exp\left\{-\left(\frac{x-\theta_2}{\theta_1}\right)^{\theta_0}\right\}$$

and parameters $\theta_0 = 2$, $\theta_1 = 2$, $\theta_2 = 0$; H_1 corresponds to gamma distribution with the density

$$f(x) = \frac{1}{\theta_1\Gamma(\theta_0)} \left(\frac{x-\theta_2}{\theta_1}\right)^{\theta_0-1} e^{-(x-\theta_2)/\theta_1}$$

and parameters $\theta_0 = 2.12154$, $\theta_1 = 0.557706$, $\theta_2 = 0$, when gamma distribution is the closest to the Weibull counterpart.

Comparing the estimates of the power for the Kuiper, Watson and Zhang tests [19] with results for Kolmogorov, Cramer–von Mises–Smirnov, and

Anderson–Darling tests [25–27], the nonparametric tests can be ordered by decrease in power as follows:

- for testing simple hypotheses with a pair "normal—logistic": $Z_C \succ Z_A \succ Z_K \succ U_n^2 \succ V_n \succ AD \succ K \succ KMS$;
- for testing simple hypotheses with a pair "Weibull—gamma": $Z_C \succ Z_A \succ Z_K \succ U_n^2 \succ V_n \succ AD \succ KMS \succ K$;
- for testing composite hypotheses with a pair "normal—logistic": $Z_A \approx Z_C \succ Z_K \succ AD \succ KMS \succ U_n^2 \succ V_n \succ K$;
- for testing composite hypotheses with a pair "Weibull—gamma": $Z_A \succ Z_C \succ AD \succ Z_K \succ KMS \succ U_n^2 \succ V_n \succ K$.

31.4 The Distribution of Statistics for Testing Composite Hypotheses

When testing composite hypotheses conditional distribution $G(S|H_0)$ of the statistic depends on several factors: the type of the observed distribution for true hypothesis H_0; the type of the estimated parameter and the number of parameters to be estimated, in some cases the parameter values (e.g., for the families of gamma and beta distributions), the method of parameter estimation. The differences between distributions of the one statistic for testing simple and composite hypotheses are very significant, so we could not neglect this fact. For example, Fig. 31.2 shows the distribution of Kuiper statistic (31.4) for testing composite hypotheses for the different distributions using maximum likelihood estimates (MLE) of the two parameters.

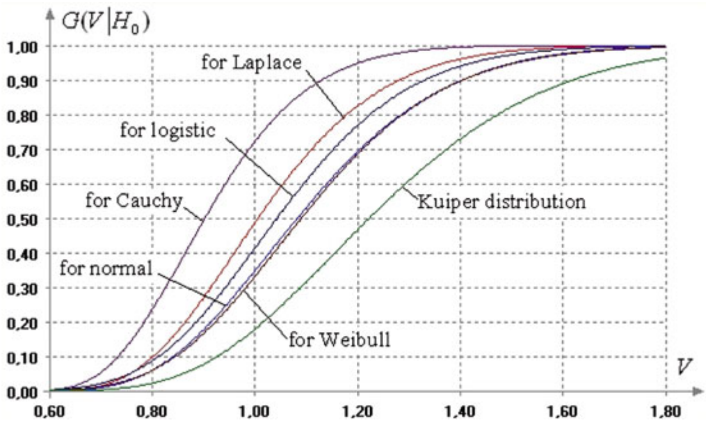

Fig. 31.2 The distribution of Kuiper statistic (31.4) for testing composite hypotheses using MLEs of the two parameters

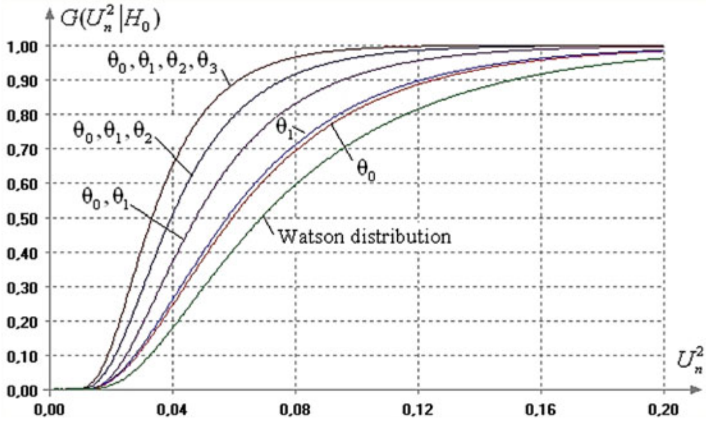

Fig. 31.3 The distribution of Watson statistic (31.5) for testing composite hypotheses using MLEs of different number of parameters of the Su-Johnson distribution

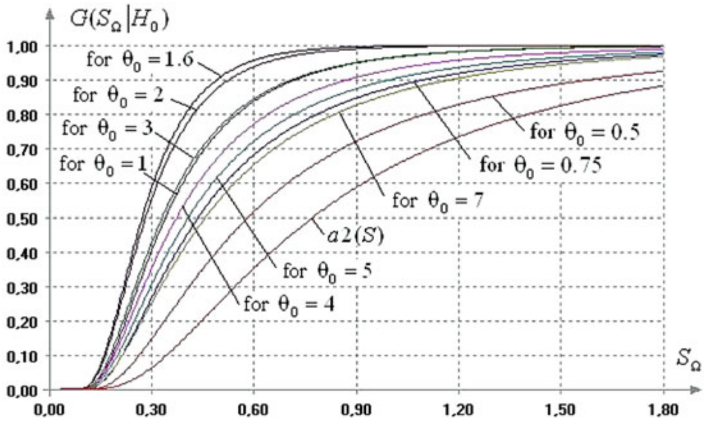

Fig. 31.4 The distribution of Anderson–Darling statistics (31.3) for testing composite hypotheses using MLEs of three parameters of the generalized normal distribution, depending on the value of the shape parameter θ_0

Figure 31.3 illustrates the dependence of the distribution of the Watson test statistic (31.5) on the type and the number of estimated parameters having as an example the *Su*-Johnson distribution with a density:

$$f(x) = \frac{\theta_1}{\sqrt{2\pi}\sqrt{(x-\theta_3)^2+\theta_2^2}} \exp\left\{-\frac{1}{2}\left[\theta_0 + \theta_1 \ln\left\{\frac{x-\theta_3}{\theta_2} + \sqrt{\left(\frac{x-\theta_3}{\theta_2}\right)^2+1}\right\}\right]^2\right\}.$$

Figure 31.4 shows the dependence of the distribution of Anderson–Darling test statistics (31.3) for testing composite hypotheses using MLEs of the three parameters of the generalized normal distribution depending on the value of the shape parameter θ_0.

The first work that initiates the study of limiting distributions of nonparametric goodness-of-fit statistics for composite hypotheses was [14]. Later, different approaches were used to solve this problem: the limit distribution was investigated by analytical methods [7–12, 30–34], the percentage points were calculated using statistical modeling [6,35,37,38], the formulas were obtained to give a good approximation for small values of the probabilities [39, 40].

In our studies [18–29] the distribution of nonparametric Kolmogorov, Cramer–von Mises–Smirnov, and Anderson–Darling tests statistics were studied using statistical modeling.

Further, based on obtained empirical distribution of statistics, we construct an approximate analytical model of statistics distributions.

The obtained models of limiting distributions and percentage points for Kuiper and Watson test statistics, which are required to test composite hypotheses (using MLEs), could be found in the paper [20] for the most often used in applications parametric distributions: Exponential, Seminormal, Rayleigh, Maxwell, Laplace, Normal, Log-normal, Cauchy, Logistic, Extreme-value (maximum), Extreme-value (minimum), Weibull, *Sb*-Johnson, *Sl*-Johnson, *Su*-Johnson.

Previously obtained similar models (and percentage points) for distributions of Kolmogorov, Cramer–von Mises–Smirnov, and Anderson–Darling test statistics (for distributions mentioned above) could be found in papers [21, 22, 24, 28].

The tables of percentage points and models of test statistics distributions were based on simulated samples of the statistics with the size $N = 10^6$. Such N makes the difference between the actual distribution $G(S|H_0)$ and empirical counterpart $G_N(S|H_0)$ that does not exceed 10^{-3}. The values of the test statistic were calculated using samples of pseudorandom values simulated for the observed distribution $F(x, \theta)$ with the size $n = 10^3$. In such a case the distribution $G(S_n|H_0)$ practically equal to the limit one $G(S|H_0)$. The given models could be used for statistical analysis if the sample sizes $n > 25$.

Unfortunately, the dependence of the nonparametric goodness-of-fit tests statistics distributions for testing composite hypotheses on the values of the shape parameter (or parameters) (see Fig. 31.4) appears to be for many parametric distributions implemented in the most interesting applications, particularly in problems of survival and reliability. This is true for families of gamma, beta distributions of the first, second, and third kind, generalized normal, generalized Weibull, inverse Gaussian distributions, and many others.

The limit distributions and percentage points for Kolmogorov, Cramer–von Mises–Smirnov, and Anderson–Darling tests for testing composite hypotheses with the family of gamma distributions were obtained in paper [22], with the inverse Gaussian distribution—in papers [29], with generalized normal distribution—in paper [23], with the generalized Weibull distribution—in paper [1]. It should be noted that the data in these papers were obtained only for a limited number of, generally, integer values of the shape parameter (or parameters).

31.5 An Interactive Method to Study Distributions of Statistics

The dependence of the test statistics distributions on the values of the shape parameter or parameters is the most serious difficulty that is faced while applying nonparametric goodness-of-fit criteria to test composite hypotheses in different applications.

Since estimates of the parameters are only known during the analysis, so the statistic distribution required to test the hypothesis could not be obtained in advance (before calculating estimates for the analyzed sample!). For criteria with statistics (31.6)–(31.8), the problem is harder as statistics distributions depend on the samples sizes. Therefore, statistics distributions of applied criteria should be obtained interactively during statistical analysis, and then should be used to make conclusions about composite hypothesis under test.

The implementation of such an interactive mode requires developed software that allows parallelizing the simulation process and taking available computing resources. While using parallel computing the time to obtain the required test statistic distribution $G_N(S_n|H_0)$ (with the required accuracy) and use it to calculate the achieved significance level $P\{S_n \geqslant S^*\}$, where S^* is the value of the statistic calculated using an original sample, is not very noticeable compared to a process of statistical analysis.

In the program system [13], an interactive method to research statistics distributions is implemented for the following nonparametric goodness-of-fit tests: Kolmogorov, Cramer–von Mises–Smirnov, Anderson–Darling, Kuiper, Watson, and three Zhang tests. Moreover, the different methods of parameter estimation could be used there.

The following example demonstrates the accuracy of calculating the achieved significance level depending on sample size N of simulated interactively empirical statistics distributions [13]. The inverse Gaussian distribution is widely used in reliability and in survival analysis [29]. In this case, the Γ-distribution (generalized gamma distribution) can be considered as the competing law.

Example. You should check the composite hypothesis that the following sample with the size $n = 100$ has the inverse Gaussian distribution with the density (31.9):

0.945	1.040	0.239	0.382	0.398	0.946	1.248	1.437	0.286	0.987
2.009	0.319	0.498	0.694	0.340	1.289	0.316	1.839	0.432	0.705
0.371	0.668	0.421	1.267	0.466	0.311	0.466	0.967	1.031	0.477
0.322	1.656	1.745	0.786	0.253	1.260	0.145	3.032	0.329	0.645
0.374	0.236	2.081	1.198	0.692	0.599	0.811	0.274	1.311	0.534
1.048	1.411	1.052	1.051	4.682	0.111	1.201	0.375	0.373	3.694
0.426	0.675	3.150	0.424	1.422	3.058	1.579	0.436	1.167	0.445
0.463	0.759	1.598	2.270	0.884	0.448	0.858	0.310	0.431	0.919
0.796	0.415	0.143	0.805	0.827	0.161	8.028	0.149	2.396	2.514
1.027	0.775	0.240	2.745	0.885	0.672	0.810	0.144	0.125	1.621

$$f(x)=\frac{1}{\theta_2}\left(\frac{\theta_0}{2\pi\left(\frac{x-\theta_3}{\theta_2}\right)^3}\right)^{1/2}\exp\left(-\frac{\theta_0\left(\left(\frac{x-\theta_3}{\theta_2}\right)-\theta_1\right)^2}{2\theta_1^2\left(\frac{x-\theta_3}{\theta_2}\right)}\right). \tag{31.9}$$

The shift parameter θ_3 is assumed to be known and equal to 0.

The shape parameters θ_0, θ_1, and the scale parameter θ_2 are estimated using the sample. The MLEs calculated using the sample above are the following: $\hat{\theta}_0 = 0.7481$, $\hat{\theta}_1 = 0.7808$, $\hat{\theta}_2 = 1.3202$. Statistics distributions of nonparametric goodness-of-fit tests depend on the values of the shape parameters θ_0 and θ_1 [46, 47], do not depend on the value of the scale parameter θ_2 and can be calculated using values $\theta_0 = 0.7481$, $\theta_1 = 0.7808$.

The calculated values of the statistics S_i^* for Kuiper, Watson, Zhang, Kolmogorov, Cramer–von Mises–Smirnov, Anderson–Darling tests and achieved significance levels for these values $P\{S \geqslant S_i^* | H_0\}$ (p-values), obtained with different accuracy of simulation (with different sizes N of simulated samples of statistics) are given in Table 31.1.

The similar results for testing goodness-of-fit of a given sample with Γ-distribution with the density:

$$f(x)=\frac{\theta_1}{\theta_3\Gamma(\theta_0)}\left(\frac{x-\theta_4}{\theta_3}\right)^{\theta_0\theta_1-1}e^{-\left(\frac{x-\theta_4}{\theta_3}\right)^{\theta_1}}$$

are given in Table 31.2. The MLEs of the parameters are $\theta_0 = 2.4933$, $\theta_1 = 0.6065$, $\theta_2 = 0.1697$, $\theta_4 = 0.10308$. In this case the distribution of the test statistic depends on the values of the shape parameters θ_0 and θ_1.

The implemented interactive mode to study statistics distributions enables to correctly apply goodness-of-fit Kolmogorov, Cramer–von Mises–Smirnov, Anderson–Darling, Kuiper, Watson, Zhang (with statistics Z_C, Z_A, Z_K) tests with calculating the achieved significance level (p-value) even in those cases when the statistic distribution for true hypothesis H_0 is unknown while testing composite hypothesis. For Zhang tests, this method allows us to test a simple hypothesis for every sample size.

Table 31.1 The achieved significance levels for different sizes N when testing goodness-of-fit with the inverse Gaussian distribution

The values of test statistics	$N = 10^3$	$N = 10^4$	$N = 10^5$	$N = 10^6$
$V_n^{\mathrm{mod}} = 1.1113$	0.479	0.492	0.493	0.492
$U_n^2 = 0.05200$	0.467	0.479	0.483	0.482
$Z_A = 3.3043$	0.661	0.681	0.679	0.678
$Z_C = 4.7975$	0.751	0.776	0.777	0.776
$Z_K = 1.4164$	0.263	0.278	0.272	0.270
$K = 0.5919$	0.643	0.659	0.662	0.662
$KMS = 0.05387$	0.540	0.557	0.560	0.561
$AD = 0.3514$	0.529	0.549	0.548	0.547

Table 31.2 The achieved significance levels for different sizes N when testing goodness-of-fit with the Γ-distribution

The values of test statistics	$N = 10^3$	$N = 10^4$	$N = 10^5$	$N = 10^6$
$V_n^{\text{mod}} = 1.14855$	0.321	0.321	0.323	0.322
$U_n^2 = 0.057777$	0.271	0.265	0.267	0.269
$Z_A = 3.30999$	0.235	0.245	0.240	0.240
$Z_C = 4.26688$	0.512	0.557	0.559	0.559
$Z_K = 1.01942$	0.336	0.347	0.345	0.344
$K = 0.60265$	0.425	0.423	0.423	0.424
$KMS = 0.05831$	0.278	0.272	0.276	0.277
$AD = 0.39234$	0.234	0.238	0.238	0.237

Acknowledgements This research has been supported by the Russian Ministry of Education and Science (project 2.541.2014K).

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