33

Inverse Gaussian Model and Its Applications in Reliability and Survival Analysis

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Abstract: Statistical properties of the parametric family of inverse Gaussian distributions are studied. Different goodness-of-fit tests for this family are compared. Some applications of the inverse Gaussian model in survival analysis and reliability are considered.

Keywords and phrases: Inverse Gaussian distribution, Generalized Weibull distribution, Hazard rate function, Goodness-of-fit test, Composite hypotheses, RRN statistic, Nonparametric goodness-of-fit tests, Kolmogorov test, Cramér–Mises–Smirnov test, Anderson-Darling test, Computer simulation, Power of the test, Reliability, Survival analysis

33.1 Introduction

Over a century the family of inverse Gaussian distributions (IGD) had attracted the attention of many researchers in several fields. The origin of this distribution goes back to the famous botanist Robert Brown (1773–1858). He interested in the study of particles motion (which now is well-known Brownian motion). In 1905, Albert Einstein derived the normal distribution as the model for Brownian motion, also in 1915 Schrödinger has obtained the distribution of first passage time as inverse Gaussian, for more details, we can see ([CF89, Ses93, Ses99]). Use of the IGD as a lifetime model is particularly appealing [GDAM97, Sin06]. The hazard rate function of the IGD has $\cap$-shape like Log-normal, generalized Weibull and Log-logistic distributions, i.e. the hazard rate of IGD is unimodal which increases from 0 to its maximum value and then decreases asymptotically to a constant. For these reasons the family of IGD is used often in reliability and survival analysis.

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33.2 The Family of the Inverse Gaussian Distributions

Let consider $X_1, X_2, \ldots, X_n$ be n independent and identically distributed random variables. We say that X_i follows the IGD and we note $X_i \sim IG(\mu, \lambda)$ if the density function is defined by

$$f(x, \theta) = \left(\frac{\lambda}{2\pi x^3} \right)^{\frac{1}{2}} \exp\left\{ -\frac{\lambda (x-\mu)^2}{2\mu^2 x} \right\}, \quad x \geqslant 0, \quad \theta = (\mu, \lambda)^T \in \mathbb{R}_+^1 \times \mathbb{R}_+^1 \subset \mathbb{R}^2, \tag{33.1}$$

where μ is the mean and λ is the shape parameter.

The density is unimodal with mode equal to

$$M_o = \mu \left\{ \left(1 + \frac{9}{4\phi^2} \right)^{\frac{1}{2}} - \frac{3}{2\phi} \right\}, \qquad \phi = \frac{\lambda}{\mu},$$

and it is easy to verify that

$$\mathbf{E}(X_i) = \mu, \quad \mathbf{Var}(X_i) = \frac{\mu^3}{\lambda}.$$

All the positive and negative moments of the IGD exist with

$$\mathbf{E}(X_i^k) = \mu^k \sum_{i=0}^{k-1} \frac{(k-1+i)!}{i!(k-1-i)!} (2\phi)^{-i}, \quad k \geqslant 1 \quad \text{and} \quad \mathbf{E}(X_i^{-k}) = \frac{E(X^{k+1})}{\mu^{2k+1}}.$$

The distribution function is

$$F(x, \theta) = \Phi\left(\sqrt{\frac{\lambda}{x}} \left(\frac{x}{\mu} - 1 \right) \right) + \exp\left(\frac{2\lambda}{\mu} \right) \Phi\left(-\sqrt{\frac{\lambda}{x}} \left(\frac{x}{\mu} + 1 \right) \right), \quad x \geqslant 0, \quad \mu, \lambda > 0. \tag{33.2}$$

The hazard rate function of IGD is

$$h(x, \theta) = \frac{\left(\frac{\lambda}{2\pi x^3} \right)^{\frac{1}{2}} \exp\{ -\frac{\lambda (x-\mu)^2}{2\mu^2 x} \}}{\Phi\left(-\sqrt{\frac{\lambda}{x}} \left(\frac{x}{\mu} - 1 \right) \right) - \exp(\frac{2\lambda}{\mu}) \Phi\left(-\sqrt{\frac{\lambda}{x}} \left(\frac{x}{\mu} + 1 \right) \right)}, \quad x \geqslant 0.$$

Since h has $\cap$-shape we may say that the family of IGD is the natural competitor of the family of Log-normal distributions (LND), the family of generalized Weibull distributions (GWD) and the family of Log-logistic distributions (LLD). We can note for example, that if we choose two densities (one from IGD and another from LND) such that the first moments are equals, then we may see that these two distributions are close to each other (Fig. 33.1).

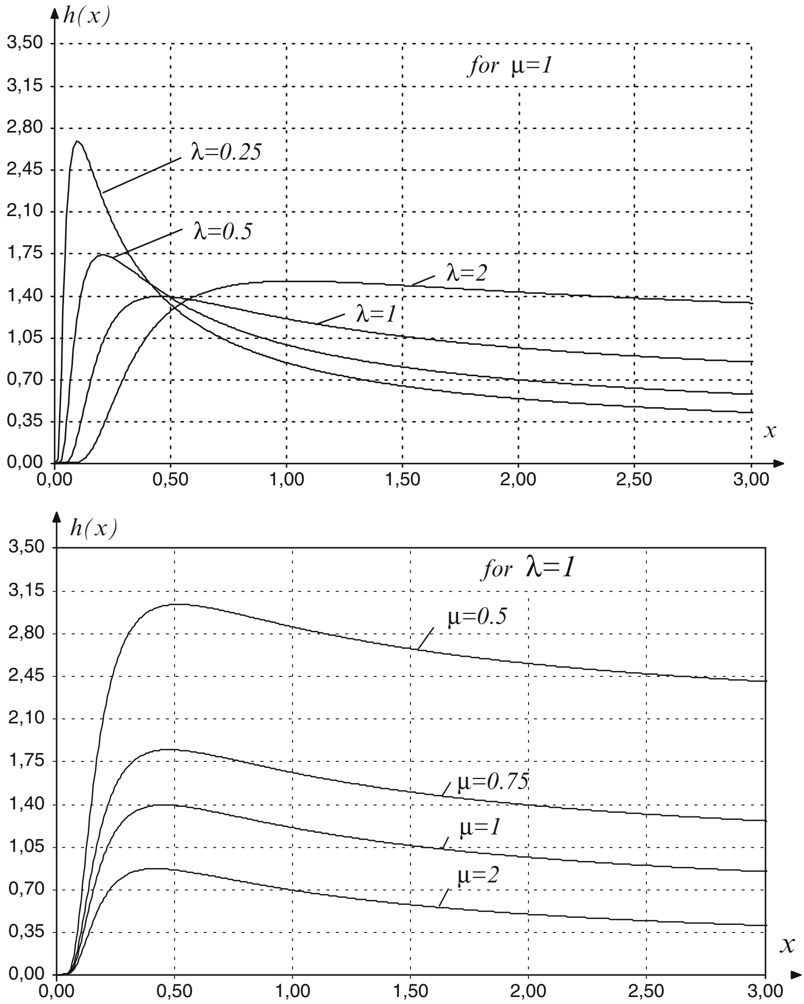

Figure 33.1. Hazard rate function of IGD

The loglikelihood function $\ell_n(\theta)$ of the sample $X_1, X_2, \ldots, X_n$ is

$$\ell_n(\theta) = \frac{n}{2}\ln\lambda - \frac{n}{2}\ln(2\pi) - \frac{3}{2}\sum_{i=1}^{n}\ln X_i - \sum_{i=1}^{n}\frac{\lambda(X_i-\mu)^2}{2\mu^2 X_i},$$

from where it follows that the bivariate statistic $T = \left(\overline{X}, V\right)^T$ is the complete minimal sufficient statistic for θ, where

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad V = \sum_{i=1}^{n}(X_i^{-1} - \frac{1}{\overline{X}}).$$

We may note here that the components $\overline{X}$ and V of the sufficient statistic are independent. It is easy to show that the maximum likelihood estimators (MLE) of μ and λ are

$$\hat{\mu}_n = \overline{X} \quad \text{and} \quad \hat{\lambda}_n = \frac{n}{\sum_{i=1}^{n}(X_i^{-1} - \frac{1}{\overline{X}})} = \frac{n}{V}$$

respectively.

The Fisher's information matrix of X_i is

$$I(\theta) = \begin{pmatrix} \frac{\lambda}{\mu^3} & 0 \\ 0 & \frac{1}{2\lambda^2} \end{pmatrix}.$$

Remark 1. The minimum variance unbiased estimators (MVUE) of μ and λ [VN93] are respectively:

$$\hat{\mu} = \overline{X} \quad \text{and} \quad \hat{\lambda} = \frac{n-3}{V}, \quad n > 3.$$

Remark 2. The MVUE of the density of IGD with unknown parameters μ and λ [VN93] is

$$\hat{f}(x, \mu, \lambda) = \begin{cases} 0, & 0 < V < C \\ \dfrac{n(n-1)\sqrt{\overline{X}^3} \quad \Gamma(\frac{n-1}{2})(V-C)^{\frac{n-4}{2}}}{\sqrt{\pi} \quad V^{\frac{n-3}{2}}\Gamma(\frac{n-2}{2})\sqrt{nx^3(n\overline{X}-x)^3}}, & V > C \end{cases}, \tag{33.3}$$

where

$$C = \frac{n(x-\overline{X})^2}{\overline{X}x(n\overline{X}-x)}, \quad n > 2.$$

Remark 3. The MVUE of the distribution function of IGD with unknown parameters μ and λ [VN93] is

$$\begin{aligned} \hat{F}(x) &= \mathbf{P}(X_i \leqslant x) \\ &= \begin{cases} 0, & x < x_1 \\ 1 - F_{n-2}(-\omega_1) + \frac{n-2}{n}\left\{1 + \frac{4(n-1)}{nV\overline{X}}\right\}^{\frac{n-3}{2}} F_{n-2}(-\omega_2), & x_1 \leqslant x \leqslant x_2 \\ 1, & x > x_2 \end{cases}, \end{aligned} \tag{33.4}$$

where

$$\omega_1 = \frac{\sqrt{n(n-2)}(x-\overline{X})}{\sqrt{V\overline{X}(n\overline{X}-x)x - n(n-\overline{X})^2}},$$

$$\omega_2 = \frac{\sqrt{n(n-2)}\left[\overline{X} + \frac{n-2}{n}x\right]}{\sqrt{V\overline{X}(n\overline{X}-x)x - n(n-\overline{X})^2}},$$

$$F_{n-2}(x) = \frac{\Gamma(\frac{n-1}{2})}{\sqrt{\pi(n-2)}\Gamma(\frac{n-2}{2})} \int_{-\infty}^{x} \left(1 + \frac{u^2}{n-2}\right)^{-\frac{n-1}{2}} \mathrm{d}u,$$

$$x_{1,2} = \frac{\overline{X}}{2(n+V\overline{X})}\left\{n(2+V\overline{X}) \pm \sqrt{4n(n-1)V\overline{X} + n^2V^2\overline{X}^2}\right\}.$$

33.3 Goodness-of-Fit Tests for the Family of IGD

Let $X_1, X_2, \ldots, X_n$ be n independent and identically distributed random variables. We consider the problem of testing the composite hypothesis H_0:

$$H_0 : \mathbf{P}(X_i \leqslant x) = F(x, \theta), \quad x \geqslant 0, \quad \theta = (\mu, \lambda)^T.$$

Goodness-of-fit tests measure the degree of agreement between the distribution of an observed data sample and a theoretical probability distribution. In all cases, a test statistic is compared with a known critical value to accept or reject the hypothesis H_0. Many statisticians have developed numerous nonparametric methods including the Chi-squared test and various empirical distribution function tests for testing H_0. The best known tests include the following one.

33.3.1 The RRN Statistic

We divide the real line into r intervals $I_1, I_2, \ldots, I_r$ by the points

$$0 = a_0 < a_1 < \cdots < a_{r-1} < a_r = +\infty,$$

$$I_i = [a_{i-1}, a_i[, \quad I_i \cap I_j = \emptyset, i \neq j, \quad \cup_{i=1}^{r} I_i = \mathbf{R}^1,$$

and we group the sample over these intervals, we obtain the vector of frequencies $\nu = (\nu_1, \nu_2, \ldots, \nu_r)^T$ and the probability vector

$$p(\theta) = (p_1(\theta), p_2(\theta), \ldots, p_r(\theta))^T, \quad \text{where} \quad p_j(\theta) = \mathbf{P}(X_1 \in I_j | H_0), \quad j = 1, 2, \ldots, r.$$

The Fisher's information matrix of the vector of frequencies ν is

$$nJ(\theta) = nB^T(\theta)B(\theta),$$

where

$$B(\theta) = \left[\frac{1}{\sqrt{p_l(\theta)}} \frac{\partial p_l(\theta)}{\partial \theta_j} \right]_{r \times 2}.$$

Let

$$q(\theta) = (\sqrt{p_1(\theta)}, \sqrt{p_2(\theta)}, \ldots, \sqrt{p_r(\theta)})^T,$$

and consider the vector

$$X_n(\theta) = \left(\frac{\nu_1 - np_1(\theta)}{\sqrt{np_1(\theta)}}, \frac{\nu_2 - np_2(\theta)}{\sqrt{np_2(\theta)}}, \ldots, \frac{\nu_r - np_r(\theta)}{\sqrt{np_r(\theta)}} \right)^T.$$

From the structure of the vector ν it follows by the multivariate Lindeberg-Levy central theorem that under H_0 and under the Cramer's regularity conditions the vector $X_n(\hat{\theta}_n)$ is $AN(\mathbf{0}_r, W(\theta))$, where $\mathbf{0}_r = (0, ..., 0)^T \in \mathbf{R}^r$, and W is the limit covariance matrix:

$$W(\theta) = I_r - q(\theta)q^T(\theta) - B(\theta)I^{-1}(\theta)B^T(\theta), \quad \text{rank} W(\theta) = r - 1.$$

For testing H_0 one may use the Chi-squared test based on the RRN statistic Y_n^2, (see, for example, [Nik73a,Nik73b,RR74,HR76,Dro88,AN94,GN96,Van98]). The RRN statistic is defined as the next quadratic form

$$Y_n^2(\hat{\theta}_n) = X_n^T(\hat{\theta}_n)W^-(\hat{\theta}_n)X_n(\hat{\theta}_n),$$

where $W^-(\theta)$ is the generalized inverse matrix of $W(\theta)$. The asymptotic behavior of the statistic $Y_n^2(\hat{\theta}_n)$ is given by the next

Theorem 1.

$$\lim_{n\to\infty} \mathbf{P}(Y_n^2(\hat{\theta}_n) \leqslant x|H_0) = \mathbf{P}(\chi_{r-1}^2 \leqslant x).$$

According to this the hypothesis H_0 must be rejected at a significance level α, if $Y_n^2(\hat{\theta}_n) > C_\alpha$, where C_α is the critical value of the test, and $C_\alpha = \chi_{r-1,\alpha}^2$ is the upper $\alpha-$ quantile of the χ^2 distribution with $r-1$ degrees of freedom.

Remark 4. For the RRN statistic, one can use the MVUE instead of the MLE.

33.3.2 The Kolmogorov, Cramér–Mises–Smirnov and Anderson–Darling Statistics

An extension of the Kolmogorov goodness-of-fit test for testing H_0 is based on the application of the random variable

$$D_n = \sup_{|n|<\infty} |F_n(x) - F(x,\theta)|, \quad \theta \in \Theta, \tag{33.5}$$

where $F_n(x)$ is the empirical distribution function. In practice it is better to use the test based on D_n with Bolshev correction [Bol87] in the form [BS83]

$$S_K = \frac{6nD_n + 1}{6\sqrt{n}}, \tag{33.6}$$

where $D_n = \max(D_n^+, D_n^-)$,

$$D_n^+ = \max_{1\leq i\leq n}\left\{\frac{i}{n} - F(x_i,\theta)\right\}, D_n^- = \max_{1\leq i\leq n}\left\{F(x_i,\theta) - \frac{i-1}{n}\right\},$$

$x_1, x_2, \ldots, x_n$ are sample values in increasing order. The distribution of statistic (33.6) obeys the Kolmogorov distribution law $K(S)$ [BS83] in the testing simple hypotheses.

The statistic of ω^2 Cramér–Mises–Smirnov goodness-of-fit test can be written as

$$S_\omega = \frac{1}{12n} + \sum_{i=1}^{n}\left\{F(x_i,\theta) - \frac{2i-1}{2n}\right\}^2, \tag{33.7}$$

and the statistic of Ω^2 Anderson–Darling test can be written in the form

$$S_\Omega = -n - 2\sum_{i=1}^{n}\left\{\frac{2i-1}{2n}\ln F(x_i,\theta) + \left(1 - \frac{2i-1}{2n}\right)\ln(1 - F(x_i,\theta))\right\}. \tag{33.8}$$

In simple hypothesis testing, the statistic (1.7) has the $a_1(S)$ distribution and statistic (1.8) has the $a_2(S)$ distribution (see [BS83]).

When testing composite hypotheses the conditional distribution law of the statistic $G(S|H_0)$ is affected by a number of factors: the form of the law $F(x,\theta)$ corresponding to the true hypothesis H_0; the method of parameter estimation and the number of estimated parameters; sometimes it is a specific value of a parameter (e.g., in the case of gamma-distribution, beta-distribution, IGD or GWD). The distinctions in the marginal statistic distributions in testing simple and composite hypotheses are so significant that we cannot neglect them.

Distribution statistic models and tables of percentage points for nonparametric goodness-of-fit tests for testing hypotheses relative to the IGD are given in [LNS09].

33.4 About Testing Hypotheses for the Inverse Gaussian Distribution

The IGD is used in reliability and survival studies along with the LND and the GWD. When constructing models of laws for really observed random variables it is sometimes difficult to choose one of the distributions mentioned above because it is complicated to distinguish these families of distributions using parametric and nonparametric goodness-of-fit tests.

Let consider an example about problems to distinguish these laws.

The sample presented (set out) below with size $n = 200$ was simulated in accordance with the IGD with parameters $\mu = 2$ and $\lambda = 2$. Pseudorandom values are given with 3 decimal digits in increasing order (by columns) in the table below (Table 33.1).

Table 33.1. Pseudorandom values

0.183	0.501	0.689	0.894	1.075	1.386	1.690	2.393	2.952	4.450
0.185	0.505	0.701	0.896	1.081	1.397	1.694	2.410	2.973	4.521
0.266	0.509	0.713	0.903	1.151	1.405	1.698	2.419	3.010	4.763
0.298	0.537	0.716	0.912	1.171	1.416	1.776	2.421	3.100	4.955
0.309	0.538	0.722	0.913	1.180	1.421	1.780	2.431	3.477	5.011
0.315	0.542	0.758	0.913	1.202	1.448	1.814	2.446	3.538	5.158
0.320	0.563	0.761	0.917	1.210	1.539	1.874	2.452	3.557	5.392
0.324	0.568	0.768	0.919	1.221	1.543	1.901	2.454	3.663	5.460
0.343	0.578	0.787	0.925	1.223	1.547	1.903	2.482	3.674	5.578
0.367	0.593	0.835	0.929	1.247	1.563	1.904	2.483	3.749	5.625
0.386	0.593	0.839	0.955	1.258	1.585	2.007	2.522	3.777	6.295
0.390	0.600	0.852	0.960	1.280	1.586	2.029	2.559	3.886	6.376
0.416	0.605	0.854	0.984	1.316	1.599	2.067	2.598	3.900	6.717
0.421	0.623	0.855	0.992	1.326	1.626	2.087	2.609	3.901	7.185
0.427	0.624	0.865	1.029	1.335	1.634	2.099	2.626	3.992	7.772
0.438	0.628	0.866	1.030	1.346	1.645	2.159	2.640	4.006	8.265
0.457	0.628	0.873	1.041	1.351	1.657	2.171	2.701	4.105	10.100
0.464	0.636	0.873	1.045	1.364	1.674	2.175	2.706	4.108	13.896
0.466	0.637	0.880	1.051	1.368	1.675	2.199	2.730	4.109	14.844
0.470	0.688	0.889	1.074	1.374	1.688	2.352	2.846	4.297	15.503

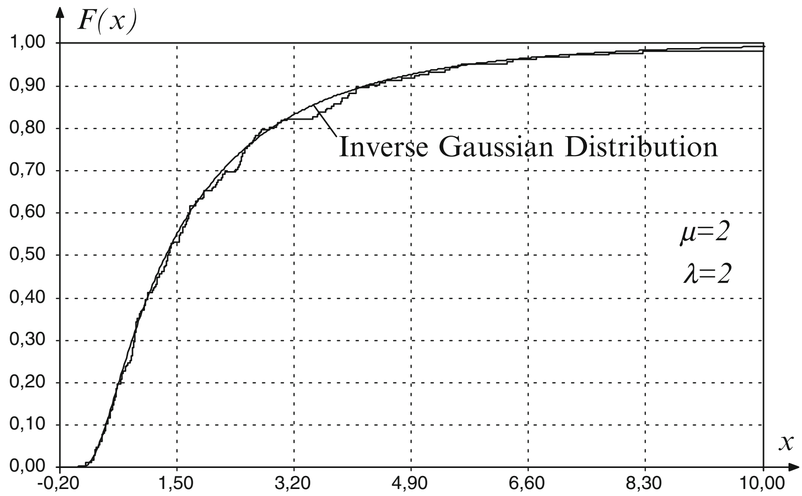

Figure 33.2. Empirical distribution and the IGD with parameters $\mu = 2$ and $\lambda = 2$

Testing a hypothesis about goodness-of-fit of the empirical distribution to the theoretical IGD has been carried out by following four criteria: RRN test [Nik73a, Nik73b, RR74] (Pearson test for simple hypothesis); Kolmogorov test; Cramér–Mises–Smirnov test; Anderson–Darling test. In case of the RRN test the number of intervals is $k = 10$. And domain of random variable is divided into equiprobable intervals (equiprobable grouping). The Neyman–Pearson intervals [GN96] are also considered in investigation of the test power. In this case, the interval bounds are chosen at the points in which the density functions corresponding to competing hypotheses intersect. The Table 1.1 contains the results of testing simple hypothesis about belonging of the sample to the IGD with parameters $\mu = 2$ and $\lambda = 2$. In the third column, there are achieved significance levels (p-values) for each test. The empirical and theoretical laws are presented in the Fig. 1.2 (Fig. 33.2).

The results of testing composite hypothesis about belonging of the sample to the IGD are presented in Table 33.3. Parameters are estimated by the maximum likelihood method ($\mu = 1.9848$ and $\lambda = 2.1119$) (by non-grouped data). In the Fig. 33.3 corresponding results are shown.

If composite hypotheses are tested the distributions of the nonparametric Kolmogorov, Cramér–Mises–Smirnov, Anderson–Darling goodness-of-fit tests depend on certain μ and λ parameter values of the IGD. In the paper [LNS09], the models of statistic distributions (and percentage points) of the nonparametric Kolmogorov, Cramér–Mises–Smirnov, Anderson–Darling goodness-of-fit tests were obtained for integer parameter values of the IGD. That is why in this case (in this situation) to obtain p-values it is necessary to model statistic distributions of the those three tests for the values $\mu = 1.9848$ and $\lambda = 2.1119$. In the Table 33.2 there are the p-values obtained basing on such modelling.

The same sample is tested for goodness-of-fit to the LND with density function

$$f(x, \mu, \sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-(lnx - \mu)^2/2\sigma^2\right), \quad x > 0, \quad \mu \in R^1, \quad \sigma > 0,$$

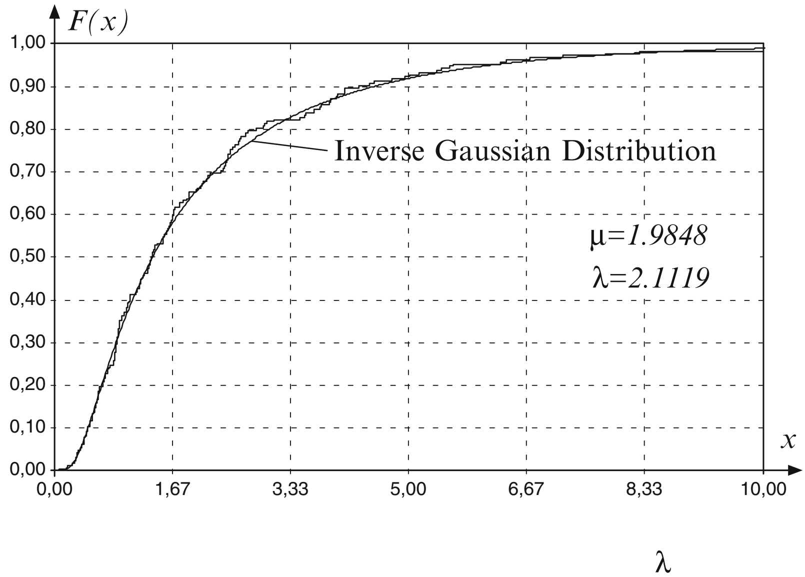

Figure 33.3. Empirical distribution and the IGD with the maximum likelihood estimates $\mu = 1.9848$ and $\lambda = 2.1119$

Table 33.2. The results of testing a simple hypothesis about belonging of the sample to the IGD with parameters $\mu = 2$ and $\lambda = 2$

Test	Statistics value	p-value
RRN test (Pearson test)	9.7000	0.3753
Kolmogorov test	0.6204	0.8362
Cramér-Mises-Smirnov test	0.0397	0.9346
Anderson-Darling test	0.2712	0.9581

and to the GWD with density function has the form

$$f(x;\theta_0,\theta_1,\theta_2) = \frac{\theta_0}{\theta_1}\theta_2{}^{\theta_0}x^{\theta_0-1}\left(1+\left(\frac{x}{\theta_2}\right)^{\theta_0}\right)^{\frac{1}{\theta_1}-1}\exp\left\{1-\left(1+\left(\frac{x}{\theta_2}\right)^{\theta_0}\right)^{\frac{1}{\theta_1}}\right\}$$

where $x > 0, \quad \theta_0, \theta_1, \theta_2 > 0$.

As you can see in the figure below three distributions (the LND, the IGD, the GWD) are close to each other. In the Fig. 33.4, there are distribution functions, in the Fig. 33.5 there are density functions and in the Fig. 33.6 there are hazard rate functions. Obviously, in practice it is difficult to make decision what law is the most appropriate. At the same time this issue plays an important role in reliability and survival studies because for close distribution functions (Fig. 33.4) we have considerable distinctions in hazard rate functions (Fig. 33.6).

In the Table 33.3, the results of testing composite hypotheses about belonging of the sample to the LND are presented. The models of statistic distributions of nonparametric goodness-of-fit tests for testing composite hypotheses included to the developed software system. These models are presented in the papers [LL09a, LL09b, LLP10].

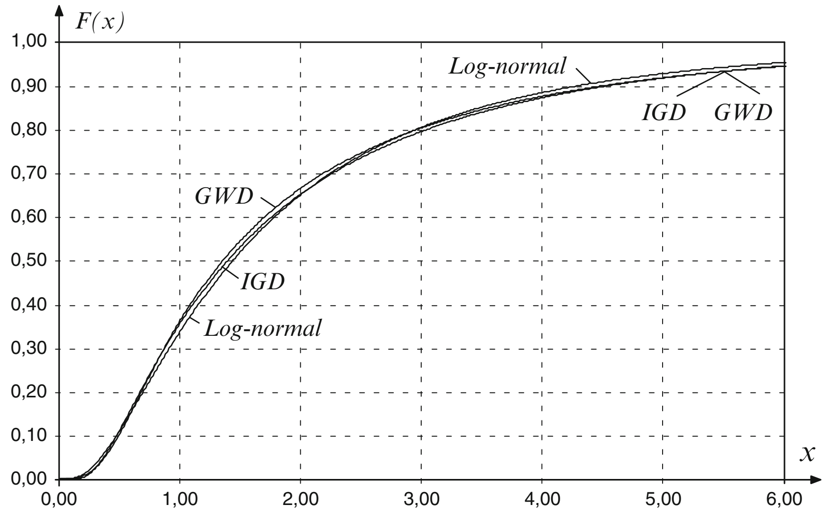

Figure 33.4. IGD ($\mu = 1.9848$, $\lambda = 2.1119$), LND ($\mu = 0.3663$, $\sigma = 0.8558$) and GWD ($\theta_0 = 3.1955$, $\theta_1 = 5.6772$, $\theta_2 = 0.5423$)

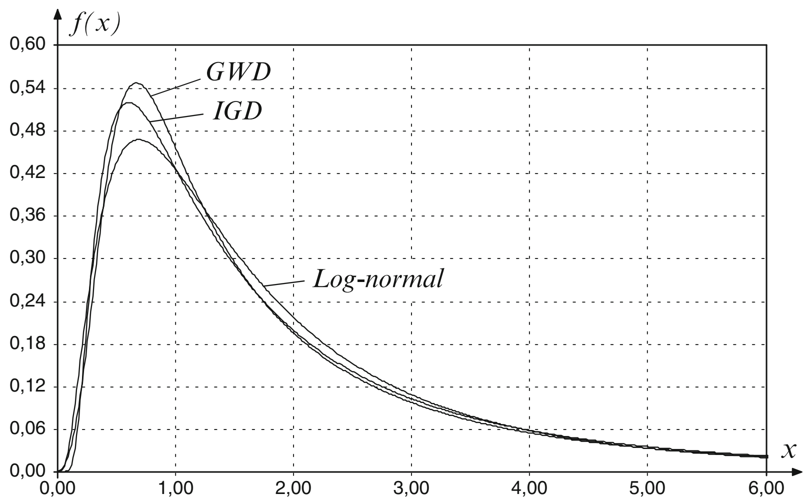

Figure 33.5. Density of the IGD ($\mu = 1.9848$, $\lambda = 2.1119$), density of the LND ($\mu = 0.3663$, $\sigma = 0.8558$) and density of the GWD ($\theta_0 = 3.1955$, $\theta_1 = 5.6772$, $\theta_2 = 0.5423$)

The Table 33.5 contains the results of testing composite hypothesis about belonging of the sample to the GWD. To obtain p-value it is necessary to model statistic distributions of the Kolmogorov, Cramér–Mises–Smirnov and Anderson–Darling test statistics with the values $\theta_0 = 3.1955$, $\theta_1 = 5.6772$ and $\theta_2 = 0.5423$. The p-value (which is presented in the Table 33.5) are obtained on the basis of modeled statistic distributions of the nonparametric goodness-of-fit tests.

The p-values obtained in testing hypothesis about the GWD practically the same as in the case of the IGD. Notice that there are no reasons to rejected hypothesis about belonging the sample to the LND.

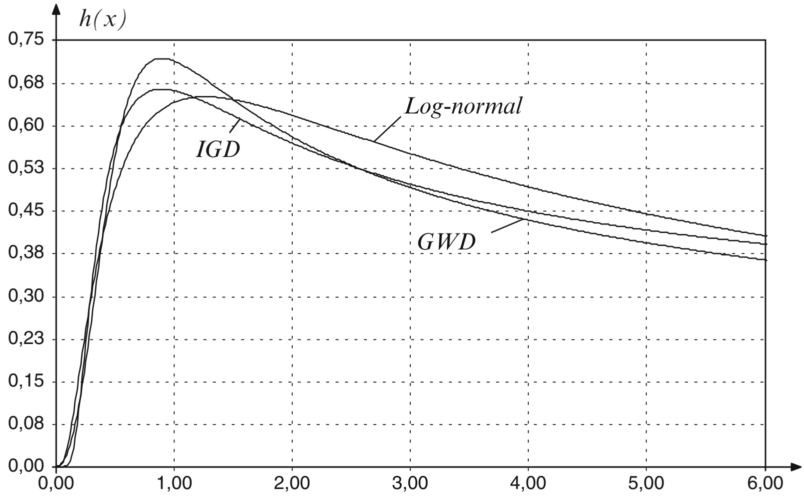

Figure 33.6. Hazard rate function of the IGD ($\mu = 1.9848$, $\lambda = 2.1119$), the LND ($\mu = 0.3663$, $\sigma = 0.8558$) and the GWD ($\theta_0 = 3.1955$, $\theta_1 = 5.6772$, $\theta_2 = 0.5423$)

Table 33.3. The results of testing composite hypothesis of goodness-of-fit to the IGD with the parameters $\mu = 1.9848$ and $\lambda = 2.1119$ estimated by the maximum likelihood method (use ungrouped observations)

Test	Statistics value	p-value
RRN test	2.6469	0.9545
Kolmogorov test	0.4875	0.9006
Cramér–Mises–Smirnov test	0.0263	0.9351
Anderson–Darling test	0.1754	0.9549

Table 33.4. The results of testing composite hypothesis of goodness-of-fit to the LND with maximum likelihood estimates (by ungrouped observations) $\mu = 0.3636$ and $\sigma = 0.8558$

Test	Statistics value	p-value
RRN test	9.1500	0.4235
Kolmogorov test	0.6524	0.4084
Cramér–Mises–Smirnov test	0.0500	0.5136
Anderson–Darling test	0.3055	0.5914

According to results in the Tables 33.3–33.5 the IGD model is more appropriate than the GWD or the LND for the sample.It is logical because the sample was modeled from the IGD. However, an isolated case does not give an opportunity (make it possible) to recognize given laws by goodness-of-fit tests. We can assess the capacity of tests to differ laws by the power of the test in testing hypothesis about belonging of the sample to the IGD, considering the LND and the GWD as the competing distributions.

The obtained power estimates of tests are presented in the Tables 33.6 and 33.7. Presented power values allow to suggest about sample sizes n due to which we can differ corresponding laws. It is evident that the LND is easier to be differed from the IGD than the GWD. At the same time it is obvious that a sure distinction requires a large value of a sample size. In particular, to get the probability of the second type error

Table 33.5. The results of testing composite hypothesis of goodness-of-fit to the GWD with maximum likelihood estimates (by ungrouped observations) $\theta_0 = 3.1955$, $\theta_1 = 5.6772$ and $\theta_2 = 0.5423$

Test	Statistics value	p-value
RRN test	6.1602	0.7238
Kolmogorov test	0.4853	0.8047
Cramér–Mises–Smirnov test	0.0257	0.8594
Anderson–Darling test	0.1636	0.9131

Table 33.6. The power of the test in testing composite hypothesis concerning the IGD ($\mu = 1.9848$, $\lambda = 2.1119$) against the LND ($\mu = 0.3636$, $\sigma = 0.8558$) as a competing hypothesis

Test	n						
	50	100	200	300	400	500	1,000
RRN test ($k = 10$, equiprobable grouping)	0.147	0.212	0.314	0.415	0.506	0.587	0.834
RRN test ($k = 5$, Neyman–Pearson classes)	0.133	0.194	0.291	0.377	0.468	0.552	0.821
Kolmogorov test	0.161	0.221	0.322	0.412	0.500	0.568	0.798
Cramér–Mises–Smirnov test	0.173	0.247	0.366	0.476	0.577	0.650	0.868
Anderson–Darling test	0.174	0.249	0.377	0.501	0.610	0.688	0.923

The probability of the first type error $\alpha = 0.1$

Table 33.7. The power of the test in testing composite hypothesis concerning the IGD ($\mu = 1.9848$, $\lambda = 2.1119$) against the GWD ($\theta_0 = 3.1955$, $\theta_1 = 5.6772$ and $\theta_2 = 0.5423$) as a competing hypothesis

Test	n						
	50	100	200	300	400	500	1,000
RRN test ($k = 10$, equiprobable grouping)	0.122	0.147	0.200	0.248	0.288	0.337	0.524
RRN test ($k = 5$, Neyman–Pearson classes)	0.132	0.145	0.179	0.211	0.226	0.242	0.325
Kolmogorov test	0.116	0.133	0.166	0.200	0.228	0.258	0.400
Cramér–Mises–Smirnov test	0.125	0.157	0.198	0.238	0.282	0.318	0.499
Anderson–Darling test	0.128	0.161	0.216	0.266	0.316	0.366	0.588

The probability of the first type error $\alpha = 0.1$

with the condition $\beta \leqslant 0.1$ for the given probability of the first type error $\alpha = 0.1$ the sample sizes $n > 2000$ are required. In this case, we test the hypothesis concerning the IGD against the competing hypothesis concerning the GWD.The power of Anderson–Darling test (Table 33.7) is 0.588 for $n = 1{,}000$ ($\beta = 0.412$). If $n = 2{,}000$ the power is 0.853 ($\beta = 0.147$), if $n = 2{,}500$ the power is 0.916 ($\beta = 0.084$). Under the same conditions to differ the IGD from the LND n should be about 1,000 (the Anderson–Darling test, Table 33.6).

33.5 Chi-Squared Goodness-of-fit Test for the Family of IGD in Case of Censored Data

In Reliability and survival analysis, we often encounter incomplete observations, and in this situation the usual methods are no longer valid. In the case of random censorship, one can use the RRN statistic $\hat{Y}_n^2$ which is well adapted for right censored data, (see [HT86]), where the Kaplan–Meier estimator $\hat{S}_n(x)$ is compared with the parametric estimator $S(x, \hat{\theta}_n)$, where $\hat{\theta}_n$ is the MLE of θ (see also [NS99]). Consider now the problem of testing the hypothesis H_0 that the data are coming from the IGD.

Under the random censorship model, we assume that the failure times $T_1, T_2, \ldots, T_n$ are nonnegative and independent. The censoring variables $C_1, C_2, \ldots, C_n$ are also nonnegative and assumed to be random sample. We observe only $X_i = min(T_i, C_i)$ and the indicator functions δ_i defined as:

$$\delta_i = \begin{cases} 1 & , \ if \quad T_i \leqslant C_i \\ 0 & , \ \text{otherwise} \end{cases} .$$

Let $S(t,\theta) = 1 - F(t,\theta), \quad \theta = (\mu, \lambda)^T$ is the survival function (or reliability function) of IGD, $f(t,\theta)$ is the density function corresponding to $F(t,\theta)$, $H(t)$ the unknown survival function of consortship and $h(t)$ the density function corresponding to $H(t)$.

The loglikelihood function is

$$\ell_n(\mu,\lambda) = \sum_{i=1}^{n} \delta_i \left\{ \frac{1}{2}\ln\lambda - \frac{1}{2}\ln 2\pi - \frac{3}{2}\ln T_i - \lambda\frac{(T_i-\mu)^2}{2\mu^2 T_i} \right\}$$

$$+ \sum_{i=1}^{n} (1-\delta_i) \ln\left\{ \Phi(A_i) - \exp\left(\frac{2\lambda}{\mu}\right) \Phi(B_i) \right\}, \tag{33.9}$$

where $A_i = -\sqrt{\frac{\lambda}{T_i}}(\frac{T_i}{\mu} - 1)$ and $B_i = -\sqrt{\frac{\lambda}{T_i}}(\frac{T_i}{\mu} + 1)$.

The score functions $U_l(\mu,\lambda), \quad l = 1, 2$ are

$$U_1(\mu,\lambda) = \frac{\partial \ell_n(\mu,\lambda)}{\partial\mu} = \frac{\lambda}{\mu^3}\sum_{i=1}^{n}\delta_i(T_i-\mu)$$

$$+ \frac{1}{\mu^2}\sum_{i=1}^{n}(1-\delta_i)\frac{\sqrt{\lambda T_i}\varphi(A_i) + \exp(\frac{2\lambda}{\mu})\left(2\lambda\Phi(B_i) - \sqrt{\lambda T_i}\varphi(B_i)\right)}{S(T_i,\mu,\lambda)},$$

$$U_2(\mu,\lambda) = \frac{\partial \ell_n(\mu,\lambda)}{\partial\lambda} = \sum_{i=1}^{n}\delta_i\left(\frac{1}{2\lambda} - \frac{(T_i-\mu)^2}{2\mu^2 T_i}\right)$$

$$+ \sum_{i=1}^{n}(1-\delta_i)\frac{\frac{1}{2\lambda}A_i\varphi(A_i) - \exp(\frac{2\lambda}{\mu})\left(\frac{2}{\mu}\Phi(B_i) + \frac{1}{2\lambda}B_i\varphi(B_i)\right)}{S(T_i,\mu,\lambda)},$$

where $\varphi(t)$ is the density function of the standard normal distribution. To have the MLE $\hat{\theta}_n$ of θ one can solve the system of equations of the score functions.

Habib and Thomas [HT86,Hjo90] have shown that $\sqrt{n}\left(\hat{S}_n(t) - S(t,\hat{\theta}_n)\right)$ converges to the Gaussian process under the hypothesis H_0.

We divide the real line into r intervals: $I_1, I_2, \ldots, I_r$ mutually disjoint by the points:

$$0 = t_0 < t_1 < \cdots < t_{r-1} < t_r = +\infty.$$

Let consider the vector

$$\hat{Z}_n = \sqrt{n}\left(\hat{S}_n - S_{\hat{\theta}_n}\right),$$

where

$$\hat{S}_n = (\hat{S}_n(t_1), \hat{S}_n(t_2), \cdots, \hat{S}_n(t_{r-1}))^T$$

and

$$S_{\hat{\theta}_n} = (S(t_1,\hat{\theta}_1), S(t_2,\hat{\theta}_2), \cdots, S(t_{r-1},\hat{\theta}_n))^T.$$

Under those assumptions:

1. $f(t,\theta)$ and $F(t,\theta)$ are twice differentiable in θ with continuous derivatives.

2. The Fisher's information matrix $I(\theta)$ is positive definite , and continuous in θ, where
$$I_{ij} = -\int \frac{\partial^2 \ln f(t,\theta)}{\partial\theta_i\theta_j} H(t) f(t,\theta) \mathrm{d}t - \int \frac{\partial^2 \ln S(t,\theta)}{\partial\theta_i\theta_j} h(t) S(t,\theta) \mathrm{d}t, \quad i,j = 1,2.$$

3. The MLE $\hat{\theta}_n$ exist and is $\sqrt{n}$- consistent estimator of θ with
$$\sqrt{n}(\hat{\theta}_n - \theta) = I^{-1} W_n + o_p(1),$$
where
$$W_n = n^{-1/2} \sum_{i=1}^{n} \frac{\partial \ln g(X_i, \delta_i, \theta)}{\partial \theta},$$
and g is the density of joint distribution of (X, δ).

Let

$$B = B(\theta) = \left[\frac{\partial F(t_i,\theta)}{\partial\theta_j}\right]_{(r-1)\times 2},$$

and

$$V = V(\theta) = \left[\mathrm{Cov}\left(Z(t_i), Z(t_j)\right)\right]_{(r-1)\times(r-1)},$$

where

$$\mathrm{Cov}\left(Z(t_i), Z(t_j)\right) = S(t_i,\theta) S(t_j,\theta) \int_0^{t_i \wedge t_j} \frac{f(t,\theta)}{H(t) F^2(t,\theta)} \mathrm{d}t.$$

To test H_0 we construct the general modified Chi-squared type of Pearson for random censored data which has the quadratic form

$$\hat{Y}_n^2(\hat{\theta}_n) = \hat{Z}_n^T \hat{\Sigma}^-(\hat{\theta}_n) \hat{Z}_n,$$

where the matrix $\hat{\Sigma}$ is the estimator of the covariance matrix Σ and Σ^- its general inverse, such that

$$\Sigma(\theta) = V(\theta) - B(\theta) I^{-1}(\theta) B^T(\theta), \quad \mathrm{rank}\Sigma = r - 1.$$

The asymptotic behavior of the statistic $\hat{Y}_n^2(\hat{\theta}_n)$ is given by the following

Theorem 2 ([HT86]).

$$\lim_{n\to\infty} \mathbf{P}(\hat{Y}_n^2(\hat{\theta}_n)) \leqslant x|H_0) = \mathbf{P}(\chi^2_{r-1} \leqslant x).$$

Note that for uncensored data, the statistic $\hat{Y}_n^2$ reduces to the considered before RRN statistic. We can also consider the case of doubly censored data (see [IL99]).

33.6 Models with Covariates Based on the Family of IGD

Failure time regression models relating the lifetime distribution to possibly time dependent external explanatory variables are considered in this section. Failure time regression models relating failure time distribution not only with external but also with internal explanatory variables will be discussed in the next section. Such models are used now not only in reliability but also in demography, dynamics of populations, gerontology, biology, survival analysis, genetics, radiobiology, biophysics, everywhere people study the problems of longevity, aging and degradation using the stochastic modeling.

In reliability, in accelerated life testing (ALT) in particular, the choice of a good regression model often is more important than in survival analysis. For example, in ALT units are tested under accelerated stresses which shorten the lifetime. Using such experiments the lifetime under the usual stress is estimated using some regression model. The values of the usual stress are often not in the range of the values of accelerated stresses, since the wide separation between experimental and usual stresses is possible, so if the model is misspecified, the estimators of survival under the usual stress may be very bad.

Let E be a set of all admission possible time-depending stresses (covariables)

$$E = \{x(\cdot) = (x_1(\cdot), x_2(\cdot), \ldots, x_m(\cdot)) : [0, +\infty) \to \mathbf{R}^m\}.$$

We denote by $T_{x(\cdot)}$ the failure time under $x(\cdot)$ and by $f_{x(\cdot)}(t)$, $S_{x(\cdot)}(t)$, $F_{x(\cdot)}(t)$ the density function , survival function and the distribution function, respectively, where

$$S_{x(\cdot)}(t) = \mathbf{P}\left(T_{x(\cdot)} \geqslant t\right) = 1 - F_{x(\cdot)}(t), \quad x(\cdot) \in E.$$

The hazard rate function of $T_{x(\cdot)}$ under $x(\cdot)$is:

$$\lambda_{x(\cdot)}(t) = \lim_{h\to 0} \frac{1}{h}\mathbf{P}\left(t \leqslant T_{x(\cdot)} < t+h \mid T_{x(\cdot)} \geqslant t\right) = -\frac{S'_{x(\cdot)}(t)}{S_{x(\cdot)}(t)}, \quad x(\cdot) \in E,$$

and we denote by:

$$\Lambda_{x(\cdot)}(t) = \int_0^t \lambda_{x(\cdot)}(u)\mathrm{d}u = -\ln(S_{x(\cdot)}(t)), \quad x(\cdot) \in E,$$

the cummulative hazard rate function of $T_{x(\cdot)}$.

33.6.1 Cox Model

The famous Cox model on E is the most popular in survival analysis. It is given in terms of the hazard rate function:

$$\lambda_{x(\cdot)} = r(x(t))\lambda_0(t), \quad x(\cdot) \in E,$$

where $\lambda_0(t)$ is the baseline hazard rate function(generally unknown) and r is a positive function often parameterized as

$$r(x) = \mathrm{e}^{\beta^T x}, \quad \beta = (\beta_1, \ldots, \beta_m)^T.$$

Let us suppose that

$$\lambda_0(t) \in \left\{ \frac{\left(\frac{\lambda}{2\pi t^3}\right)^{\frac{1}{2}} \ \exp\{-\frac{\lambda(t-\mu)^2}{2\mu^2 t}\}}{\Phi\left(-\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu}-1\right)\right) - \ \exp(\frac{2\lambda}{\mu})\Phi\left(-\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu}+1\right)\right)}, \ t \geqslant 0, \ \mu > 0, \ \lambda > 0 \right\}.$$

In such, we have the so-called parametric Cox Inverse Gaussian model, where the vector β expresses the effect of covariate $x(\cdot)$ on the distribution of the lifetime $T_{x(\cdot)}$. Let consider the hypothesis

$$H_0 : \beta = 0.$$

Under H_0 there is no influence of covariates on distribution of the lifetime $T_{x(\cdot)} = T$, and in this case (under H_0) the lifetime T follows an inverse Gaussian distribution. So to test H_0 one can use, for example, the Chi-squared test exposed before.

33.6.2 AFT Model

The term accelerated life testing applies that the type of study where failure times can be accelerated by applying higher "stress" to the component or system reliability, and higher stress may bring quicker failure. For example, some component may fail quicker at a higher temperature; however, it may have a long lifetime at normal temperatures. At normal stress conditions, the time required may be too large for its reliability estimation which may be tested under higher stress factors terminating the experiment. We look at case where the hazard rate function has the $\cap$-shape.

We consider some applications of the family of IGD as the baseline survivals in the construction of the accelerated failure time (AFT) model which is very natural competitor of lognormal and generalized Weibull distributions.

The AFT model holds on E, (see [BN02]), if there exists on E a positive function r and a survival function S_0 such that:

$$S_{x(\cdot)}(t) = S_0\left(\int_0^t r(x(u))\mathrm{d}u\right), \quad t \geqslant 0, \quad x(\cdot) \in E, \tag{33.10}$$

where S_0 is the baseline survival function. In term of hazard rate function, the expression (1.10) holds if and only if there exists on E a positive function r and on $[0, +\infty)$ a positive function q such that

$$\alpha_{x(\cdot)}(t) = r(x(t)q(S_{x(\cdot)}(t)), \quad x(\cdot) \in E.$$

In parametric case S_0 belongs to a parametric family, and $r(x)$ is often parameterized as in the Cox model: $r(x) = e^{\beta^T x}$.

If we consider the inverse Gaussian distribution as models for the baseline survival function S_0 such that

$$S_0(t) = \Phi\left(-\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu}-1\right)\right) - \exp\left(\frac{2\lambda}{\mu}\right)\Phi\left(-\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu}+1\right)\right), \quad t \geqslant 0, \quad \mu, \lambda > 0,$$

we obtain the AFT_{IG} model.

Let consider the hypothesis

$$H_0 : \beta = 0.$$

Under H_0 there is no influence of covariates on distribution of the lifetime $T_{x(\cdot)} = T$, and under H_0 the distribution of the lifetime T is inverse Gaussian, and one can test it using the chi-square test, for example.

Remark 5. We say that the distribution of $T_{x(\cdot)}$ belongs to the class GPH (Generalized Proportional Hazards) on E (see [BN02]) if its hazard rate function is given by formula

$$\lambda_{x(\cdot)} = r(x(t))q(\Lambda_{x(\cdot)})\lambda_0(t), \quad x(\cdot) \in E,$$

where $q(\cdot)$ is a positive function on R^1_+. Model GPH is very interesting since it generalizes the famous Cox and AFT models on E. It is evident that in some situations we need to test the Cox model against the AFT model, and in this situation it is very interesting to apply the GPH model.

33.6.3 Inverse Gaussian Family and Analysis of Redundant System

Now we shall consider, following the papers of Bagdonavičius, Masiulaityle and Nikulin [BMN08a, BMN08b, BMN09, BMN10], one example of the parametric estimation of redundant system reliability when the distribution of the failure time in "hot" and "warm" conditions belongs to the IGD. Let consider redundant system $S(1, m-1)$ with one principal main unit operating in "hot" and $(m-1)$ stand-by units operating in "warm" conditions. The problem is to estimate the parameters of the redundant system, using failure data of two groups of units, when we suppose that switching from warm to hot does not cause shock or damage to units.

Denote by T_1, F_1 and f_1 the failure time, the cumulative distribution function and the probability density function of the main unit. The failure times of the stand-by units denote by $T_2, \ldots, T_m$. In "hot" conditions their distribution functions are also F_1. In "warm" conditions the distribution function of T_i is F_2 and the density function is f_2, $i = 2, \ldots, m$. If a stand-by unit is switched to "hot" conditions, its cumulative distribution function is different from F_1 and F_2.

The failure time of the system $S(1, m-1)$ is

$$T^{(m)} = T_1 \vee T_2 \vee \cdots \vee T_m.$$

Denote by K_j and k_j the distribution function and the density function of $T^{(j)}$, respectively, $(j = 2, \dots, m)$, $K_1 = F_1$, $k_1 = f_1$. The distribution function K_j can be written in terms of the distribution function K_{j-1} and F_1:

$$K_j(t) = \mathbf{P}(T^{(j)} \leqslant t) = \int_0^t \mathbf{P}(T_j \leqslant t | T^{(j-1)} = y) \mathrm{d}K_{j-1}(y).$$

The "fluent switch on" hypothesis H_0 formulated by Bagdonavičius et al. [BMN08a, BMN08b] states that

$$f_{T_j|T^{(j-1)}=y}(t) = \begin{cases} f_2(t) & \text{if} \quad t \leqslant y, \\ f_1(t + g(y) - y) & \text{if} \quad t > y; \end{cases},$$

where $g(y) = F_1^{-1}(F_2(y))$. This model implies that

$$K_j(t) = \int_0^t F_1(t + g(y) - y) \mathrm{d}K_{j-1}(y), \quad j = 2, ..., m.$$

So the distribution function K_m of the system with $m - 1$ stand-by units is defined recurrently by the last formula. We consider here the situation when the distribution of units functioning in "warm" and "hot" conditions differ only in scale, i.e. we suppose that $g(y) = ry$ and hence $F_2(t) = F_1(rt)$ for all $t \geqslant 0$ and some $r > 0$. In such a case the cumulative distribution function of units functioning in "hot" and "warm" conditions mostly belong to the same parametric classes of distributions, for example, to the family of IGD. If the cumulative distribution function of units belongs to the family of IGD then

$$S_1(t, \mu, \lambda) = 1 - F_1(t, \mu, \lambda) = \Phi\left(-\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu} - 1\right)\right) - \exp\left(\frac{2\lambda}{\mu}\right) \Phi\left(-\sqrt{\frac{\lambda}{t}}\left(\frac{t}{\mu} + 1\right)\right).$$

Parametric Estimators of the Parameter $\gamma = (r, \mu, \lambda)^T$

Suppose that the following data are available:

(a) Complete ordered sample $T_{11}, \dots, T_{1n_1}$ of size n_1, T_{1i} is the failure time of units tested in 'hot' condition;
(b) Complete ordered sample $T_{21}, \dots, T_{2n_2}$ of size n_2, T_{2i} is the failure time of units tested in 'warm' condition.

Let $\gamma = (r, \mu, \lambda)^T$, the MLE $\hat{\gamma} = (\hat{r}, \hat{\mu}, \hat{\lambda})^T$ of the parameter γ maximizes the loglikelihood function

$$\ell(r, \mu, \lambda) = \frac{n}{2} \ln \lambda - \frac{n}{2} \ln(2\pi) - \frac{n_2}{2} \ln r + \frac{\lambda n}{\mu} - \frac{3}{2} \sum_{i=1}^{n_1} \ln(T_{1i}) - \frac{3}{2} \sum_{i=1}^{n_2} \ln(T_{2i})$$

$$- \frac{\lambda}{2\mu^2} \sum_{i=1}^{n_1} T_{1i} - \frac{\lambda}{2} \sum_{i=1}^{n_1} T_{1i}^{-1} - \frac{\lambda r}{2\mu^2} \sum_{i=1}^{n_2} T_{2i} - \frac{\lambda}{2r} \sum_{i=1}^{n_2} T_{2i}^{-1},$$

where $n = n_1 + n_2$.

The score functions are

$$\frac{\partial \ell}{\partial r} = -\frac{n_2}{2r} - \frac{\lambda}{2\mu^2}\sum_{i=1}^{n_2} T_{2i} + \frac{\lambda}{2r^2}\sum_{i=1}^{n_2} T_{2i}^{-1},$$

$$\frac{\partial \ell}{\partial \mu} = -\frac{\lambda n}{\mu^2} + \frac{\lambda}{\mu^3}\sum_{i=1}^{n_1} T_{1i} + \frac{\lambda r}{\mu^3}\sum_{i=1}^{n_2} T_{2i},$$

$$\frac{\partial \ell}{\partial \lambda} = n\left(\frac{1}{2\lambda} + \frac{1}{\mu}\right) - \frac{1}{2\mu^2}\sum_{i=1}^{n_1} T_{1i} - \frac{1}{2}\sum_{i=1}^{n_1} T_{1i}^{-1} - \frac{r}{2\mu^2}\sum_{i=1}^{n_2} T_{2i} - \frac{1}{2r}\sum_{i=1}^{n_2} T_{2i}^{-1}.$$

To find the estimator $\hat{\gamma}$ one can solve the system formed by equalizing the score functions to zero.

The Fisher information matrix is

$$I(\gamma) = \begin{pmatrix} \frac{(\mu+2\lambda)n_2}{2\mu r^2} & -\frac{\lambda n_2}{r\mu^2} & -\frac{n_2}{2r\lambda} \\ -\frac{\lambda n_2}{r\mu^2} & \frac{\lambda n}{\mu^3} & 0 \\ -\frac{n_2}{2r\lambda} & 0 & \frac{n}{2\lambda^2} \end{pmatrix},$$

The inverse of this matrix is

$$I^{-1}(\gamma) = \begin{pmatrix} \frac{2nr^2\mu}{n_1 n_2(\mu+2\lambda)} & \frac{2r\mu^2}{n_1(\mu+2\lambda)} & \frac{2r\mu\lambda}{n_1(\mu+2\lambda)} \\ \frac{2r\mu^2}{n_1(\mu+2\lambda)} & \frac{(n_1\mu+2n\lambda)\mu^3}{n_1 n(\mu+2\lambda)\lambda} & \frac{2n_2\mu^2\lambda}{n_1 n(\mu+2\lambda)} \\ \frac{2r\mu\lambda}{n_1(\mu+2\lambda)} & \frac{2n_2\mu^2\lambda}{n_1 n(\mu+2\lambda)} & \frac{2(n\mu+2n_1\lambda)\lambda^2}{n_1 n(\mu+2\lambda)} \end{pmatrix}.$$

Taking $j = 2$, the cumulative distribution function $K_2(t)$ is estimated by

$$\hat{K}_2(t) = \int_0^t \sqrt{\frac{\hat{\lambda}}{2\pi y^3}}\exp\left\{-\frac{\hat{\lambda}(y-\hat{\mu})^2}{2\hat{\mu}^2 y}\right\}\Phi\left(\sqrt{\frac{\hat{\lambda}}{t+\hat{r}y-y}}\left(\frac{t+\hat{r}y-y}{\hat{\mu}} - 1\right)\right)\mathrm{d}y$$

$$+\int_0^t \sqrt{\frac{\hat{\lambda}}{2\pi y^3}}\exp\left\{-\frac{\hat{\lambda}(y-\hat{\mu})^2}{2\hat{\mu}^2 y} + \frac{2\hat{\lambda}}{\hat{\mu}}\right\}\Phi\left(-\sqrt{\frac{\hat{\lambda}}{t+\hat{r}y-y}}\left(\frac{t+\hat{r}y-y}{\hat{\mu}} + 1\right)\right)\mathrm{d}y.$$

Now using the results of Bagdonavičius et al. [BMN08a, BMN08b, BMN09, BMN10] we can construct the asymptotic $1-\alpha$ confidence interval $(\underline{K}_2(t), \overline{K}_2(t))$ for $K_2(t)$, with

$$\underline{K}_2(t) = \left(1 + \frac{1-\hat{K}_2(t)}{\hat{K}_2(t)}\exp\left\{\frac{\hat{\sigma}_{\hat{K}_2} z_{1-\alpha/2}}{\sqrt{\hat{K}_2(t)(1-\hat{K}_2(t))}}\right\}\right)^{-1},$$

$$\overline{K}_2(t) = \left(1 + \frac{1-\hat{K}_2(t)}{\hat{K}_2(t)}\exp\left\{-\frac{\hat{\sigma}_{\hat{K}_2} z_{1-\alpha/2}}{\sqrt{\hat{K}_2(t)(1-\hat{K}_2(t))}}\right\}\right)^{-1},$$

here

$$\hat{\sigma}^2_{\hat{K}_2(t)} = C_2^T(t,\hat{\gamma})I^{-1}(\hat{\gamma})C_2(t,\hat{\gamma}),$$

where

$$C_2(t,\gamma) = (C_{21}(t,\gamma), C_{22}(t,\gamma), C_{23}(t,\gamma))^T,$$

$$C_{21}(t,\gamma) = \int_0^t \frac{\partial F_1}{\partial r}(t+ry-y,\mu,\lambda)\mathrm{d}F_1(y,\mu,\lambda),$$

$$C_{22}(t,\gamma) = \int_0^t \frac{\partial F_1}{\partial \mu}(t+ry-y,\mu,\lambda)\mathrm{d}F_1(y,\mu,\lambda) + F_1(y,\mu,\lambda)\mathrm{d}\left(\frac{\partial F_1}{\partial \mu}(y,\mu,\lambda)\right),$$

$$C_{23}(t,\gamma) = \int_0^t \frac{\partial F_1}{\partial \lambda}(t+ry-y,\mu,\lambda)\mathrm{d}F_1(y,\mu,\lambda) + F_1(y,\mu,\lambda)\mathrm{d}\left(\frac{\partial F_1}{\partial \lambda}(y,\mu,\lambda)\right),$$

and $z_{1-\alpha/2}$ is the $(1-\alpha/2)$-quantile of the standard normal distribution.

At the end of this section we note that using the results of [BMN10] it is easy to estimate the parameter γ when the data are censored.

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References

[AN94] Aguirre, N., Nikulin, M.: Chi-squared goodness-of-fit test for the family of logistic distributions. Kybernetika, **V.30**, 214–222, (1994).

[BMN08a] Bagdonavičius, V., Masiulaityle, I., Nikulin, M.: Statistical analysis of redundant system with "warm" stand-by units. Stochastics: An International Journal of Probability and Stochastical Processes, **80**, N 2–3, 115–128, (2008a).

[BMN08b] Bagdonavičius, V., Masiulaityle, I., Nikulin, M.: Statistical analysis of redundant system with one stand-by unit. In: Mathematical Methods in Survival Analysis, Reliability and Quality of Life. (Huber, C., Limnios, N., Balakrishnan, N., Messbah, M., Nikulin, M. (Eds)), ISTE & Wiley: London, 189–202, (2008b).

[BMN09] Bagdonavičius, V., Masiulaityle, I., Nikulin, M.: Advances in Degradation Modeling: Applications to Reliability, Survival Analysis, and Finance. (Nikulin, M., Limnios, N., Balakrishnan, N., Kahle, W., Huber, C. (Eds)), Springer Verlag: Boston, 293–310, (2009).

[BMN10] Bagdonavičius, V., Masiulaityle, I., Nikulin, M.: Parametric estimation of redundant system reliability from censored data. In: Mathematical and Statistical Methods in Reliability. (Balakrishnan, N., Nikulin, M., Rykov, V. (Eds)), Springer Verlag: Boston, (2010).

[BS83] Bolshev, L.N., Smirnov, N.V.: Tables of Mathematical Statistics. Science: Moscow, (1983). [in Russian].

[Bol87] Bolshev, L.N.: On the question on testing some composite statistical hypotheses. In: Theory of Probability and Mathematical Statistics. Selected Works. Nauka: Moscow, 5–63, (1987).

[CF89] Chhikara, R.S., Folks, J.L.: The Inverse Gaussian Distribution. Marcel Dekker: New York, (1989).

[Dro88] Drost, F.: Asymptotics for generalized Chi-squared goodness-of-fit tests. CWI Tracs, Amsterdam: Centre for Mathematics and Computer Sciences, **V.48**, (1988).

[GN96] Greenwood, P.E., Nikulin, M.S.: A Guide to Chi-Squared Testing. Wiley: New York, (1996).

[GDAM97] Gunes, H., Dietz, D.C., Auclai, P.F., Moor, A.H.: Modified goodness-of-fit tests for the inverse Gaussian distribution. Computational Statistics and Data analysis, **V.24**, 63–77, (1997).

[HT86] Habib, M.G., Thomas, D.R.: Chi-squared goodness-of-fit tests for randomly censored data. Annals of Statistics, **V.14**, 759–765, (1986).

[Hjo90] Hjort, N.L.: Goodness-of-fit tests in models for life history data based on cumulative hazard rates. Annals of Statistics, **V.18**, 1221–1258, (1990).

[HR76] Hsuan, T.A., Robson, D.S.: The χ^2-goodness-of-fit tests with moment type estimator. Communications in Statistics – Theory and Methods, **V.16**, 1509–1519, (1976).

[IL99] Ionescu, D.C., Lmmnios, N. (Eds) Statistical and Probabilistic Models in Reliability, Birkhauser: Boston, (1999).

[LNS09] Lemeshko, S.B., Nikulin, M.S., Saaidia, N.: Simulation of the statistics distributions and power of the goodness-of-fit tests in composite hypotheses testing rather inverse Gaussian distribution. Proceedings of the 6th St. Petersburg Workshop on Simulation, St. Petersburg, June 28–July 4. VVM comm. Ltd., St. Petersburg, **V.1**, 323–328, (2009).

[LL09a] Lemeshko, B.Yu., Lemeshko, S.B.: Distribution models for nonparametric tests for fit in verifying complicated hypotheses and maximum-likelihood estimators. Part 1. Measurement Techniques, **V.52**, N 6, 555–565, (2009).

[LL09b] Lemeshko, B.Yu., Lemeshko, S.B.: Models for statistical distributions in nonparametric fitting tests on composite hypotheses based on maximum-likelihood estimators. Part II. Measurement Techniques, **V.52**, N 8, 799–812, (2009).

[LLP10] Lemeshko, B.Yu., Lemeshko, S.B., Postovalov, S.N.: Statistic distribution models for some nonparametric goodness-of-fit tests in testing composite hypotheses. Communications in Statistics – Theory and Methods, **V.39**, Issue 3, 460-471, (2010).

[Nik73a] Nikulin, M.S.: On a Chi-squared test for continuous distributions. Theory Probability and its Applications. **V.18**, N 3, 638–639, (1973).

[Nik73b] Nikulin, M.S.: Chi-squared test for continuous distributions with shift and scale parameters. Theory Probability and its Applications, **V.18**, N 3, 559–568, (1973).

[NS99] Nikulin, M.S., Solev, V.N.: Chi-squares goodness-of-fit test for doubly censored data with applications in survival analysis and reliability. In: Statistical and Probabilistic Models in Reliability. (Ionescu, D.C., Limnios, N. (Eds)). Birkhauser: Boston, 101–112, (1999).

[RR74] Rao, K.C., Robson, D.S.: A Chi-squared statistic for goodness-of-fit tests within the exponential family. Communication Statistics. **V.3**, 1139–1153, (1974).

[Ses93] Seshadri, V.: The Inverse Gaussian Distribution: A Case Study in Exponential Families. Clarendon Press: Oxford, (1993).

[Ses99] Seshadri V.: The Inverse Gaussian Distribution: Statistical Theory and Applications. Springer: New York, (1999).

[Sin06] Singpurwalla, N.D.: Reliability and Risk. Wiley: Chichester, (2006).

[Van98] Van Der Vaart, A.W.: Asymptotic Statistics. Cambridge Series in Statistics and probabilistic Mathematics. Cambridge University Press: Cambridge, (1998).

[VN93] Voinov, V., Nikulin, M.: Unbiased Estimators and their Applications, V.1 Univariate case. Kluwer: Dordrecht, (1993).