Chapter 5

Interactive Investigation of Statistical Regularities in Testing Composite Hypotheses of Goodness of Fit

In this chapter[1], a "real-time" ability to simulate and research distributions of test statistics in the course of testing a complex hypothesis of goodness of fit for distributions with estimated parameters is implemented by means of parallel computing. It makes it possible to make correct statistical inferences even in those situations when the distribution of test statistic is unknown before the testing procedure starts.

5.1. Introduction

In composite hypotheses testing in the form $H_0 : F(x) \in \{F(x, \theta), \theta \in \Theta\}$, when an estimate $\hat{\theta}$ of scalar or vector distribution parameter θ is calculated from sample under testing, the non-parametric goodness-of-fit Kolmogorov (K), ω^2 Cramér–von Mises–Smirnov (CMS) and Ω^2 Anderson–Darling (AD) tests lose their property of being distribution free.

The value:

$$D_n = \sup_{|x|<\infty} |F_n(x) - F(x, \theta)|$$

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is used in the Kolmogorov test as a distance between empirical and theoretical laws (where $F_n(x)$ is an empirical distribution function, and n is the size of the sample). When testing hypotheses, this statistic should be used with Bolshev's correction [BOL 87] in the form [BOL 83]:

$$S_K = \frac{6nD_n + 1}{6\sqrt{n}}, \qquad [5.1]$$

where $D_n = \max(D_n^+, D_n^-)$,

$$D_n^+ = \max_{1 \le i \le n} \left\{ \frac{i}{n} - F(x_i, \theta) \right\}, D_n^- = \max_{1 \le i \le n} \left\{ F(x_i, \theta) - \frac{i-1}{n} \right\},$$

n is the size of the sample, and $x_1, x_2, \ldots, x_n$ are sample values in an increasing order. The statistic [5.1] in testing a simple hypothesis follows the Kolmogorov distribution $K(s) = \sum\limits_{k=-\infty}^{\infty} (-1)^k e^{-2k^2 s^2}$.

In the ω^2 CMS test, we use a statistic in the form:

$$S_\omega = \frac{1}{12n} + \sum_{i=1}^{n} \left\{ F(x_i, \theta) - \frac{2i-1}{2n} \right\}^2, \qquad [5.2]$$

and in a test of Ω^2 AD type [AND 52, AND 54], a statistic in the form:

$$S_\Omega = -n - 2\sum_{i=1}^{n} \left\{ \frac{2i-1}{2n} \ln F(x_i, \theta) + \left(1 - \frac{2i-1}{2n}\right) \ln(1 - F(x_i, \theta)) \right\}. \qquad [5.3]$$

In testing a simple hypothesis, [5.2] has the distribution with CDF $a1(\cdot)$ (see [BOL 83]):

$$\begin{aligned} a1(s) \quad = \quad & \frac{1}{\sqrt{2s}} \sum_{j=0}^{\infty} \frac{\Gamma(j+1/2)\sqrt{4j+1}}{\Gamma(1/2)\Gamma(j+1)} \exp\left\{ -\frac{(4j+1)^2}{16s} \right\} \\ & \times \left\{ I_{-\frac{1}{4}} \left[\frac{(4j+1)^2}{16s} \right] - I_{\frac{1}{4}} \left[\frac{(4j+1)^2}{16s} \right] \right\}, \end{aligned}$$

where $I_{-\frac{1}{4}}(\cdot)$ and $I_{\frac{1}{4}}(\cdot)$ are the modified Bessel functions:

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}, \quad |z| < \infty, \quad |\arg z| < \pi,$$

and [5.3] has the distribution with CDF $a2(\cdot)$ [BOL 83]:

$$\begin{aligned} a2(s) \quad &= \quad \frac{\sqrt{2\pi}}{s} \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(j+1/2)(4j+1)}{\Gamma(1/2)\Gamma(j+1)} \exp\left\{-\frac{(4j+1)^2\pi^2}{8s}\right\} \times \\ &\times \int_0^{\infty} \exp\left\{\frac{s}{8(y^2+1)} - \frac{(4j+1)^2\pi^2 y^2}{8s}\right\} dy. \end{aligned}$$

5.2. Distributions of the test statistics in the case of testing composite hypotheses

Let us denote the distribution of a statistic S when hypothesis H_0 is true as $G(S|H_0)$. In our case, we will consider the statistics of Kolmogorov S_K, CMS S_ω and AD S_Ω in place of S. In composite hypothesis testing, $G(S|H_0)$ is affected by a number of factors: the form of the distribution $F(x, \theta)$ that corresponds to the true hypothesis H_0; types and number of parameters to be estimated; the method of parameter estimation; value of a specific parameter (e.g. for gamma distribution, inverse Gaussian distribution (IGD), generalized Weibull distribution (GWD), generalized Gaussian distribution (GGD) and beta-distribution families).

The investigation of statistic distributions of the non-parametric goodness-of-fit tests for composite hypotheses began from the paper [KAC 55]. Then, various approaches to solve this problem were used [DAR 55, DAR 57, DUR 73, DUR 75, DUR 76, GIH 53, MAR 78, MAR 11, PEA 72, STE 70, STE 74, CHA 81, TYU 84a, TYU 84b, DZH 82, NIK 92a, NIK 92b].

In our research [LEM 98, LEM 01, LEM 02, LEM 04, LEM 07a, LEM 07b, LEM 09a, LEM 09b, LEM 10a, LEM 10b, LEM 10c, LEM 11a, LEM 11b], the distributions of statistics of the non-parametric goodness-of-fit tests have been investigated by means of methods of statistical simulation, and approximate models of the distributions have been found on the basis of the obtained empirical distributions. The most complete list of the constructed models of distributions of statistics and tables of percentage points for the non-parametric goodness-of-fit tests is provided in [LEM 09a], [LEM 09b], [LEM 10c], [LEM 11a], [LEM 11b] and [LEM 11f]. These models and tables are applicable when maximum likelihood (ML) estimators (MLEs) are used.

The distributions of statistics of non-parametric goodness-of-fit tests are affected by a number of factors: the form of the observed law $F(x, \theta)$ that corresponds to the true hypothesis H_0; types and number of parameters to be estimated; the method of parameter estimation. In these cases, there are no impediments for studying test statistic distributions by means of statistical simulation and further construction of approximate models for them when testing composite hypothesis [LEM 10a, LEM 10b, LEM 10c].

Complications arise when distributions $G(S|H_0)$ of non-parametric goodness-of-fit tests depend on the value of specific parameter(s) of $F(x, \theta)$ (for gamma distribution, GGD, IGD, GWD and beta-distribution families).

The existing dependence on parameters values should not be neglected. For example, in composite hypotheses testing subject to GWD with the density function:

$$\begin{aligned} f(x;\theta_0,\theta_1,\theta_2) &= \frac{\theta_0}{\theta_1}{\theta_2}^{\theta_0} x^{\theta_0-1}\left(1+\left(\frac{x}{\theta_2}\right)^{\theta_0}\right)^{\frac{1}{\theta_1}-1} \\ &\quad \times \exp\left(1-\left(1+\left(\frac{x}{\theta_2}\right)^{\theta_0}\right)^{\frac{1}{\theta_1}}\right), \end{aligned} \qquad [5.4]$$

where $x \geq 0$ and θ_0, θ_1, θ_2 are positive, the limiting distributions of statistics of the non-parametric goodness-of-fit tests depend on value of the form parameter θ_1. In Figures 5.1 and 5.2 we can see the behavior of the distribution of statistic [5.3] in testing composite hypotheses for family [5.4]. In the case when three parameters are estimated by the ML method (Figure 5.1), we can see the following: when the value of the shape parameter increases up to $\theta_1 \approx 2$, the distribution $G(S|H_0)$ shifts to the left. With θ_1 continuing to increase further, the distribution $G(S|H_0)$ shifts into the opposite direction. In a case of two parameters estimated by MLE (Figure 5.2), we can see the following: as value of θ_1 increases, the distribution $G(S|H_0)$ shifts to the right.

Percentage points have been obtained by simulations, and models of marginal statistic distributions of Kolmogorov, CMS and AD tests have been computed for the following values of shape parameter θ_1: 0.5, 1.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0. The points and models are presented in Tables 5.1–5.6. The distribution parameters were estimated with ML method.

Distributions $G(S|H_0)$ of the Kolmogorov, the CMS and the AD statistics are best of all approximated by the family of the type III beta-distributions with the density function:

$$B_3(\theta_0,\theta_1,\theta_2,\theta_3,\theta_4) = \frac{\theta_2^{\theta_0}}{\theta_3 B(\theta_0,\theta_1)} \frac{\left(\frac{x-\theta_4}{\theta_3}\right)^{\theta_0-1}\left(1-\frac{x-\theta_4}{\theta_3}\right)^{\theta_1-1}}{\left[1+(\theta_2-1)\frac{x-\theta_4}{\theta_3}\right]^{\theta_0+\theta_1}},$$

or by the family of the *Sb*-Johnson distributions:

$$Sb(\theta_0,\theta_1,\theta_2,\theta_3) = \frac{\theta_1\theta_2}{(x-\theta_3)(\theta_2+\theta_3-x)} \exp\left\{-\frac{1}{2}\left(\theta_0-\theta_1 \ln\frac{x-\theta_3}{\theta_2+\theta_3-x}\right)^2\right\}.$$

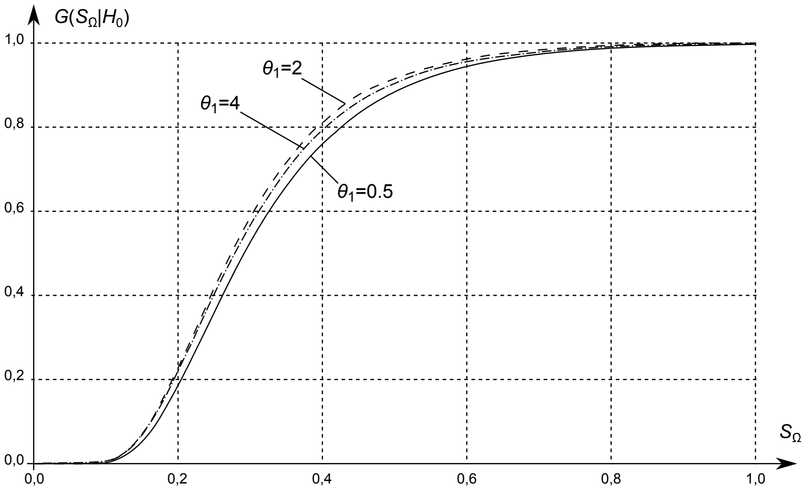

Figure 5.1. *Distributions of the Anderson–Darling statistic in testing goodness of fit of family [5.4]. ML method is used to estimate parameters θ_0, θ_1 and θ_2*

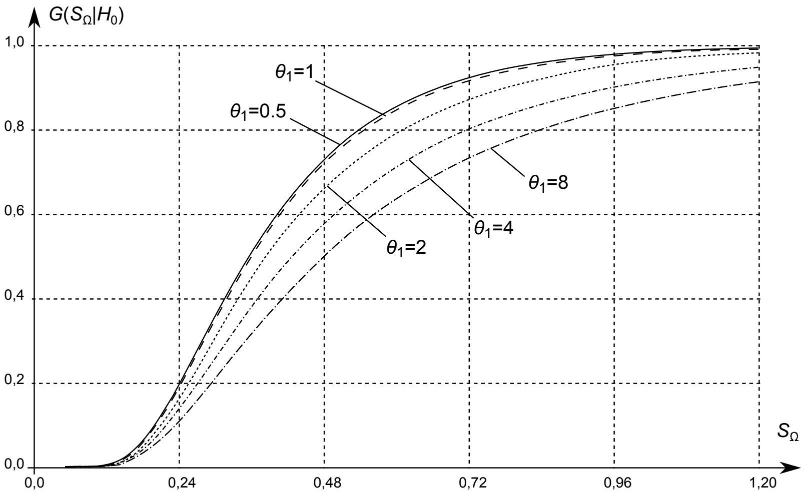

Figure 5.2. *Distributions of the Anderson–Darling statistic in testing goodness of fit of family [5.4]. ML method is used to estimate parameters θ_0 and θ_1*

The tables of percentage points and the models of distributions of statistics have been constructed from simulated samples of statistics of the size $N = 10^6$. Under this size, the deviation the empirical cumulative distribution function (CDF) $G_N(S|H_0)$

from the theoretical counterpart is less than 10^3. The values of statistics of goodness-of-fit tests have been calculated using samples of size $n = 10^3$ of pseudorandom variables following the given $F(x, \theta)$. For this value of n, CDF. $G(S_n|H_0)$ of statistics is almost the same as the marginal CDF. $G(S|H_0)$.

Parameters estimated	Percentage points			Model
	0.9	0.95	0.99	
The Kolmogorov test				
θ_0	1.001	1.102	1.309	B_3(6.5294, 6.8315, 3.5901, 2.0446,0.2801)
θ_1	1.084	1.199	1.427	B_3(5.4860, 5.9744, 3.4348, 2.1402, 0.3000)
θ_2	1.038	1.144	1.360	B_3(4.7833, 6.1285, 3.0596, 2.0214, 0.3200)
θ_0, θ_1	0.849	0.922	1.071	B_3(6.2332, 6.0259,2.8200, 1.3000, 0.2800)
θ_0, θ_2	0.837	0.909	1.054	*Sb*(2.1787, 1.8756, 1.5259, 0.2567)
θ_1, θ_2	0.848	0.922	1.076	*Sb*(2.4861, 1.8758, 1.7026, 0.2664)
$\theta_0, \theta_1, \theta_2$	0.780	0.845	0.979	*Sb*(2.3507, 1.9291, 1.4629, 0.2495)
The Cramer–von Mises–Smirnov test				
θ_0	0.181	0.232	0.359	B_3(5.1297, 2.5959, 22.9591,0.8000, 0.0081)
θ_1	0.227	0.296	0.466	B_3(7.4650, 2.6576, 44.4162, 1.3633, 0.0000)
θ_2	0.198	0.255	0.395	B_3(5.4489, 2.7019, 31.5609, 1.1500, 0.0062)
θ_0, θ_1	0.110	0.135	0.192	B_3(6.3779, 4.6451, 27.3376, 1.0000, 0.0050)
θ_0, θ_2	0.106	0.129	0.183	*Sb*(3.7541, 1.5434, 0.5800, 0.0058)
θ_1, θ_2	0.112	0.138	0.200	B_3(10.3369, 4.0734, 25.8270, 0.5802, 0.0000)
$\theta_0, \theta_1, \theta_2$	0.086	0.103	0.145	B_3(6.7252, 4.6508, 16.7920, 0.4800, 0.0050)
The Anderson–Darling test				
θ_0	1.125	1.415	2.140	B_3(4.9800, 4.1685, 17.0454, 7.1000, 0.0500)
θ_1	1.279	1.625	2.478	B_3(4.7602, 5.1000, 9.8527, 6.8675, 0.0000)
θ_2	1.157	1.454	2.186	B_3(3.0331, 4.0598, 9.3429, 5.9880, 0.1000)
θ_0, θ_1	0.673	0.806	1.120	B_3(5.7172, 5.0419, 10.1641, 3.0044, 0.0550)
θ_0, θ_2	0.655	0.781	1.079	*Sb*(3.8953, 1.6481, 3.5052, 0.0513)
θ_1, θ_2	0.743	0.902	1.290	*Sb*(4.1462, 1.6136, 4.6254, 0.0535)
$\theta_0, \theta_1, \theta_2$	0.523	0.617	0.839	*Sb*(3.9313, 1.6905, 2.7078, 0.0530)

Table 5.1. *Percentage points and models of the limiting distributions of statistics of the non-parametric goodness-of-fit tests when ML method is used for parameter estimation (θ_1 = 0.5)*

The most serious impediment to a complete solution of the problem of testing composite hypotheses by means of the non-parametric goodness-of-fit tests is that the distributions of the test statistics depend on specific values of shape parameters of the observed laws. In [LEM 07a, LEM 09a, LEM 09b, LEM 10a, LEM 10b, LEM 10c, LEM 11a and LEM 11b], models of distributions of statistics were obtained for a limited set of combinations of (integer) values of shape parameters (for gamma distribution, two-sided exponential law, IGD and beta-distribution families). It is unrealistic to build the models for an infinite set of combinations of the parameter values.

Parameters estimated	Percentage points			Model
	0.9	0.95	0.99	
The Kolmogorov test				
θ_0	1.181	1.316	1.585	B_3(6.9734, 4.8247, 5.3213, 2.3800, 0.2690)
θ_1	1.083	1.196	1.425	B_3(4.6425, 6.6688, 2.8491, 2.2246, 0.3200)
θ_2	0.994	1.092	1.290	B_3(6.2635, 7.1481, 3.2059, 2.0000, 0.2800)
θ_0, θ_1	0.874	0.954	1.117	*Sb*(2.4299, 1.8866, 1.7504, 0.2598)
θ_0, θ_2	0.823	0.893	1.033	B_3(5.8989, 7.5040, 2.4180, 1.3724, 0.2800)
θ_1, θ_2	0.815	0.883	1.023	*Sb*(2.4499, 1.9720, 1.6016, 0.2486)
$\theta_0, \theta_1, \theta_2$	0.758	0.820	0.946	*Sb*(2.3012, 1.9386, 1.3863, 0.2464)
The Cramer–von Mises–Smirnov test				
θ_0	0.320	0.431	0.706	B_3(2.2422, 2.2970, 16.4663, 1.6500, 0.0130)
θ_1	0.227	0.295	0.464	B_3(5.3830, 2.6954, 40.5199, 1.6450, 0.0050)
θ_2	0.174	0.221	0.336	B_3(3.6505, 3.2499, 16.5445, 1.0000, 0.0100)
θ_0, θ_1	0.117	0.144	0.209	*Sb*(3.8667, 1.4603, 0.7583, 0.0059)
θ_0, θ_2	0.102	0.123	0.174	B_3(12.2776, 4.1107, 27.2069, 0.4875, 0.0000)
θ_1, θ_2	0.103	0.127	0.182	B_3(4.7144, 4.6690, 10.8816, 0.5261, 0.0059)
$\theta_0, \theta_1, \theta_2$	0.080	0.097	0.135	*Sb*(4.1842, 1.6587, 0.4794, 0.0061)
The Anderson–Darling test				
θ_0	1.724	2.280	3.639	B_3(4.8106, 2.6855, 35.5593,11.8700, 0.0500)
θ_1	1.275	1.617	2.468	B_3(3.6999, 3.9108, 16.4841, 9.0300, 0.0740)
θ_2	1.056	1.314	1.953	B_3(4.9871, 4.1479, 16.5432, 6.4500, 0.0600)
θ_0, θ_1	0.687	0.827	1.161	B_3(4.6368, 6.6727, 7.1680, 3.6356, 0.0521)
θ_0, θ_2	0.633	0.753	1.037	B_3(3.0467, 5.9239, 5.0944, 2.7870, 0.1000)
θ_1, θ_2	0.696	0.842	1.194	B_3(6.9638, 4.5238, 17.7792, 3.8000, 0.0522)
$\theta_0, \theta_1, \theta_2$	0.494	0.582	0.786	*Sb*(3.9578, 1.6861, 2.5760, 0.0547)

Table 5.2. *Percentage points and models of the limiting distributions of statistics of the non-parametric goodness-of-fit tests when ML method is used for parameter estimation ($\theta_1 = 1$)*

In this chapter, a “real-time” ability is implemented by the use of parallel computing to simulate and research the distributions of test statistics in the case of testing a composite goodness-of-fit hypothesis (for distributions with estimated parameters). It makes it possible to make correct statistical inferences even in those situations when the distribution of the test statistic is unknown (before the testing procedure starts).

Parameters estimated	Percentage points			Model
	0.9	0.95	0.99	
The Kolmogorov test				
θ_0	1.150	1.279	1.538	B_3(5.0155, 5.4869, 3.3992, 2.2476, 0.3000)
θ_1	1.084	1.199	1.427	B_3(5.4860, 5.9744, 3.4348, 2.1402,0.3000)
θ_2	0.892	0.968	1.121	B_3(4.6527, 7.8624, 1.8636, 1.4770, 0.3110)
θ_0, θ_1	0.992	1.095	1.301	B_3(37.6836, 9.6249, 24.7703, 4.2400, 0.1000)
θ_0, θ_2	0.823	0.895	1.041	B_3(6.6694, 6.5961, 3.0264,1.3700, 0.2650)
θ_1, θ_2	0.807	0.877	1.020	B_3(5.3859, 8.4947, 2.3199, 1.4900, 0.2850)
$\theta_0, \theta_1, \theta_2$	0.751	0.811	0.932	B_3(5.7236, 7.0743, 2.3212, 1.1488, 0.2714)
The Cramer–von Mises–Smirnov test				
θ_0	0.286	0.383	0.620	*Sb*(3.4745, 1.1215, 2.1611, 0.0065)
θ_1	0.227	0.296	0.466	B_3(8.0420, 2.6222, 50.1417, 1.3950, 0.0000)
θ_2	0.135	0.167	0.240	B_3(9.8988, 3.6331, 27.2342, 0.6611, 0.0000)
θ_0, θ_1	0.167	0.215	0.334	*Sb*(3.6343, 1.2549, 1.1752, 0.0074)
θ_0, θ_2	0.100	0.121	0.172	B_3(4.9109, 4.8805, 11.3991, 0.5400, 0.0058)
θ_1, θ_2	0.099	0.121	0.174	B_3(9.7955, 5.0455, 35.0176, 0.9000, 0.0000)
$\theta_0, \theta_1, \theta_2$	0.078	0.094	0.133	B_3(4.2414, 3.7719, 8.6839, 0.2744, 0.0087)
The Anderson–Darling test				
θ_0	1.556	2.040	3.234	B_3(4.3943, 2.4670, 38.0035, 10.7000, 0.0900)
θ_1	1.279	1.625	2.478	B_3(5.3689, 3.2667, 21.3222, 6.8675, 0.0535)
θ_2	0.900	1.096	1.572	B_3(3.5132, 4.3501, 8.8168, 4.2500, 0.1000)
θ_0, θ_1	0.871	1.088	1.623	B_3(5.6254, 3.7452, 20.0868, 4.9237, 0.0588)
θ_0, θ_2	0.619	0.737	1.015	B_3(7.1939, 6.8828, 3.2613, 1.5626, 0.2598)
θ_1, θ_2	0.640	0.769	1.072	B_3(30.1793 ,4.4373, 60.5986, 3.2000, 0.0000)
$\theta_0, \theta_1, \theta_2$	0.483	0.568	0.773	B_3(5.2772, 4.4958, 7.9102, 1.5891, 0.0664))

Table 5.3. *Percentage points and models of the limiting distributions of statistics of the non-parametric goodness-of-fit tests when the ML method is used for parameter estimation (θ_1= 3)*

5.3. Testing composite hypotheses in "real-time"

In this chapter, we propose an approach that is based upon the software, that is being developed by authors, and the use of simulation [LEM 11e, LEM 11]. Computational processes in the simulation of distributions of statistics of various tests can be parallelized rather easily by the use of available resources of nearby computer network. This makes it possible to dramatically reduce the time required to simulate (study) an unknown distribution $G(S|H_0)$ of statistic. Statistical analysis is carried out by the scheme in Figure 5.3 in the case of the use of non-parametric goodness-of-fit tests for testing composite hypotheses in regard to the distributions, for which the statistic distributions depend on parameter values. Such an approach was used in [LEM 10b] and [LEM 11d]. Here, the study of $G(S|H_0)$ is carried out in "real time" of testing the hypothesis [LEM 11c].

Parameters estimated	Percentage points			Model
	0.9	0.95	0.99	
The Kolmogorov test				
θ_0	1.131	1.256	1.506	B_3(5.0752, 5.5757, 3.3089, 2.1797, 0.3000)
θ_1	1.084	1.199	1.427	B_3(3.8892, 6.2974, 2.5413, 2.1402, 0.3400)
θ_2	0.890	0.966	1.119	*Sb*(2.1569,1.8555, 1.6361, 0.2661)
θ_0, θ_1	1.024	1.133	1.352	B_3(14.6423, 5.3789, 9.0355, 2.1287, 0.2000)
θ_0, θ_2	0.839	0.914	1.068	B_3(5.1515, 6.1071, 2.8573, 1.3900, 0.3000)
θ_1, θ_2	0.833	0.909	1.065	B_3(7.3590, 7.0743, 3.0755, 1.4500, 0.2450)
$\theta_0, \theta_1, \theta_2$	0.769	0.834	0.970	B_3(4.0431, 7.9330, 1.6664, 1.2059, 0.3007)
The Cramer–von Mises–Smirnov test				
θ_0	0.269	0.359	0.578	*Sb*(3.4774, 1.1443, 1.9761, 0.0066)
θ_1	0.227	0.296	0.466	B_3(7.2936, 2.6369, 40.7763, 1.2800, 0.0000)
θ_2	0.135	0.166	0.240	B_3(6.9544, 4.2952, 17.0098, 0.7100, 0.0000)
θ_0, θ_1	0.186	0.242	0.379	B_3(10.0457, 2.7234, 74.1688, 1.4000, 0.0000)
θ_0, θ_2	0.104	0.127	0.182	B_3(10.3993, 4.2771, 25.5455, 0.5600, 0.0000)
θ_1, θ_2	0.104	0.128	0.186	B_3(5.2006, 4.4814, 13.7165, 0.5770, 0.0050)
$\theta_0, \theta_1, \theta_2$	0.082	0.099	0.141	B_3(4.3747, 3.2066, 9.2236, 0.2479, 0.0088)
The Anderson–Darling test				
θ_0	1.484	1.934	3.047	B_3(12.5725, 2.7914, 75.0000, 9.6500, 0.0000)
θ_1	1.279	1.625	2.478	B_3(6.9691, 2.9121, 32.3978, 6.8675, 0.0535)
θ_2	0.895	1.090	1.556	B_3(16.0792, 4.1280, 41.0115, 4.9000, 0.0000)
θ_0, θ_1	0.958	1.213	1.838	B_3(5.9821, 3.4306, 23.7037, 5.4000, 0.0500)
θ_0, θ_2	0.632	0.754	1.043	B_3(19.4692, 4.7303, 32.4566, 2.8950, 0.0000)
θ_1, θ_2	0.645	0.776	1.087	B_3(19.2831, 4.8148, 37.5002, 3.4100, 0.0000)
$\theta_0, \theta_1, \theta_2$	0.496	0.587	0.805	B_3(5.9771,4.3144,9.7987,1.7085,0.0619)

Table 5.4. *Percentage points and models of the limiting distributions of statistics of the non-parametric goodness-of-fit tests when ML method is used for parameter estimation (θ_1= 4)*

When the composite hypothesis $H_0 : F(x) \in \{F(x, \theta), \theta \in \Theta\}$ is tested by an existing sample $x_1, x_2, \ldots, x_n$, the parameter vector estimate $\hat{\theta}$ for the distribution $F(x, \theta)$ is found in accordance with the selected method (MLE, in the case). Then, the value of statistic S^* of the goodness-of-fit test in use is calculated in accordance with the estimate $\hat{\theta}$ found. To make an inference on whether to reject or to accept the hypothesis H_0 under test, it is necessary to know the distribution $G(S|H_0)$ of the test statistic that corresponds to the parameter value $\hat{\theta}$.

After that, a statistical simulation procedure is started that results in obtaining empirical distribution $G_N(S_n|H_0)$ of the test statistic for the corresponding sample size n, the given number of simulations N and the parameters vector $\theta = \hat{\theta}$ of $F(x, \theta)$. We can find an estimate of an achieved significance level $P\{S_n > S^*\}$ or estimates of percentage points by the use of empirical distribution $G_N(S_n|H_0)$. The hypothesis is not rejected if $P\{S_n > S^*\} > \alpha$, where α is the given type I error probability.

Parameters estimated	Percentage points			Model
	0.9	0.95	0.99	
The Kolmogorov test				
θ_0	1.104	1.223	1.461	B_3(5.0453,5.6018,3.3300, 2.1145, 0.3100)
θ_1	1.084	1.199	1.427	B_3(5.3655, 6.0543, 3.3092, 2.1402, 0.3000)
θ_2	0.955	1.047	1.231	B_3(8.8643, 20.9468, 7.9001, 9.1000, 0.2300)
θ_0, θ_1	1.066	1.180	1.409	*Sb*(2.4625, 1.7390, 2.3814, 0.2668)
θ_0, θ_2	0.895	0.980	1.153	B_3(4.2520, 7.5684, 2.1829, 1.6786, 0.3100)
θ_1, θ_2	0.902	0.991	1.169	B_3(4.5096, 5.6482, 3.0218, 1.6000, 0.3100)
$\theta_0, \theta_1, \theta_2$	0.839	0.918	1.079	B_3(8.5291, 6.5470, 4.4062, 1.6000, 0.2400)
The Cramer–von Mises–Smirnov test				
θ_0	0.246	0.323	0.516	B_3(7.5042, 2.4317, 48.3146, 1.4000, 0.0000)
θ_1	0.227	0.296	0.466	B_3(6.2641, 2.8729, 33.7742, 1.3750, 0.0000)
θ_2	0.156	0.196	0.290	B_3(4.1621, 3.9072, 14.0226, 0.8986, 0.0059)
θ_0, θ_1	0.213	0.278	0.441	*Sb*(3.4488, 1.2020, 1.4196, 0.0061)
θ_0, θ_2	0.122	0.151	0.223	B_3(7.9405, 3.8743, 23.4697, 0.6700, 0.0000)
θ_1, θ_2	0.124	0.155	0.234	B_3(7.5192, 4.0675, 25.1497, 0.7945, 0.0000)
$\theta_0, \theta_1, \theta_2$	0.010	0.123	0.181	B_3(5.5784, 3.2913, 17.0579, 0.4290, 0.0067)
The Anderson–Darling test				
θ_0	1.380	1.778	2.770	*Sb*(3.7593, 1.3295, 9.6362, 0.0552)
θ_1	1.279	1.625	2.478	B_3(4.8031, 3.4732, 17.6302, 6.8675, 0.0535)
θ_2	0.972	1.193	1.724	B_3(3.4890, 4.7102, 10.2828, 5.9597, 0.0800)
θ_0, θ_1	1.111	1.420	2.184	B_3(5.4232, 3.1894, 26.3229, 6.7000, 0.0539)
θ_0, θ_2	0.692	0.835	1.178	B_3(11.9769, 4.7144, 19.2233, 3.0000, 0.0000)
θ_1, θ_2	0.701	0.852	1.225	B_3(22.3537, 4.1744, 51.2639, 3.6000, 0.0000)
$\theta_0, \theta_1, \theta_2$	0.563	0.677	0.955	B_3(8.0353, 3.9949, 18.1724, 2.4000, 0.0500)

Table 5.5. *Percentage points and models of the limiting distributions of statistics of the non-parametric goodness-of-fit tests when ML method is used for parameter estimation (θ_1= 7)*

The value of N defines the required accuracy of simulation of $G(S_n|H_0)$. However, the time spent for simulation increases along with the growth of N; therefore, we can determine N during parallelization of simulation process on the basis of available computer resources (number of processors and cores) that could be used for the problem under solution.

The probability that elements of $\hat{\theta}$ are integer is zero. Thus, we should cautiously use models and percentage points of test statistic distributions for values of parameters close to integer values provided in [LEM 07a, LEM 09a, LEM 09b, LEM 10a, LEM 10b and LEM 10c] as, with interpolation applied, results obtained can be far from the true distribution $G(S|H_0)$ with the given $\hat{\theta}$.

Parameters estimated	Percentage points			Model
	0.9	0.95	0.99	
The Kolmogorov test				
θ_0	1.100	1.218	1.454	B_3(8.0781, 4.8128, 5.8094, 2.0960, 0.2735)
θ_1	1.084	1.199	1.428	*Sb*(2.4326, 1.7778, 2.3797, 0.2673)
θ_2	0.978	1.074	1.266	B_3(8.4485, 5.1812, 5.5890, 1.8364, 0.2700)
θ_0, θ_1	1.072	1.186	1.417	B_3(5.7833, 6.1641, 3.2903, 2.1269, 0.2699)
θ_0, θ_2	0.911	0.999	1.179	*Sb*(2.6863, 1.8734, 2.0545, 0.2559)
θ_1, θ_2	0.929	1.012	1.198	*Sb*(2.6357, 1.8244, 2.0497, 0.2612)
$\theta_0, \theta_1, \theta_2$	0.863	0.948	1.117	B_3(11.1281, 6.1031, 6.0962, 1.7021, 0.2200)
The Cramer–von Mises–Smirnov test				
θ_0	0.242	0.319	0.507	B_3(4.5895, 2.5584, 15.2153, 0.8500, 0.0000)
θ_1	0.228	0.296	0.467	B_3(6.0112, 2.5379, 23.0339, 0.8900, 0.0000)
θ_2	0.166	0.209	0.314	B_3(5.8877, 4.0329, 23.3907, 1.2150, 0.0000)
θ_0, θ_1	0.217	0.284	0.450	*Sb*(3.4552, 1.1997, 1.4606, 0.0061)
θ_0, θ_2	0.128	0.160	0.239	*Sb*(4.6035, 1.4434, 1.3182, 0.0060)
θ_1, θ_2	0.131	0.165	0.252	*Sb*(4.4612, 1.4003, 1.3183, 0.0059)
$\theta_0, \theta_1, \theta_2$	0.107	0.134	0.201	B_3(6.9845, 2.7596, 2.6920, 0.4000, 0.0060)
The Anderson–Darling test				
θ_0	1.363	1.752	2.722	B_3(5.6824, 4.0065, 18.9636, 8.4000, 0.0000)
θ_1	1.279	1.624	2.477	*Sb*(3.4000, 1.3163, 7.4752, 0.0535)
θ_2	1.005	1.240	1.799	B_3(3.4843, 5.3032, 9.1592, 6.2767, 0.0800)
θ_0, θ_1	1.139	1.457	2.243	B_3(6.3736, 2.8599, 35.0312, 6.7458, 0.0538)
θ_0, θ_2	0.713	0.864	1.227	B_3(4.6820, 5.7296, 7.8880, 3.4597, 0.0523)
θ_1, θ_2	0.722	0.881	1.275	B_3(4.3613, 6.0352, 7.6499, 3.8116, 0.0520)
$\theta_0, \theta_1, \theta_2$	0.589	0.714	1.009	B_3(7.9147, 4.0088, 21.3294, 2.9120, 0.0500)

Table 5.6. *Percentage points and models of the limiting distributions of statistics of the non-parametric goodness-of-fit tests when ML method is used for parameter estimation ($\theta_1 = 8$)*

Let us consider an example where a composite hypothesis is tested in regard to the IGD with the density function:

$$f(x) = \left(\frac{\theta_1}{2\pi x^3}\right)^{1/2} \exp\left(-\frac{\theta_1 (x-\theta_0)^2}{2\theta_0^2 x}\right).$$

In this case, distributions $G(S|H_0)$ of the non-parametric tests depend on specific values of θ_0 and θ_1.

$x_1, x_2, \ldots, x_n$

↓

Calculation of $\hat{\theta}$ for $F(x, \theta)$

↓

Calculation of the test statistic S^*

↓

Simulation of $G_N(S_n | H_0)$ for $H_0: F(x) \in \{F(x,\theta),\ \theta \in \Theta\}$ when $\theta_{TRUE} = \hat{\theta}$

↓

Calculation of $P\{S_n > S^*\}$

Figure 5.3. *Testing the composite hypothesis* $H_0 : F(x) \in \{F(x, \theta), \theta \in \Theta\}$

The following is the sample under analysis:

0.278	0.633	0.928	1.078	1.334	1.937	2.297	2.630	3.554	5.674
0.312	0.686	0.933	1.080	1.497	1.965	2.362	2.919	3.593	5.989
0.358	0.716	0.936	1.089	1.612	1.991	2.364	2.995	3.948	6.284
0.361	0.776	0.938	1.113	1.671	2.012	2.417	3.002	3.996	6.863
0.362	0.777	0.956	1.119	1.680	2.026	2.467	3.120	4.053	7.580
0.374	0.789	0.996	1.159	1.687	2.027	2.566	3.149	4.141	7.644
0.403	0.796	1.038	1.165	1.731	2.069	2.577	3.166	4.363	7.874
0.590	0.805	1.053	1.166	1.735	2.146	2.599	3.224	4.597	9.236
0.597	0.822	1.060	1.192	1.763	2.210	2.621	3.278	5.022	11.704
0.599	0.849	1.066	1.245	1.898	2.213	2.628	3.528	5.201	20.069

ML estimates of parameters calculated are $\hat{\theta_0} = 2.4706$ and $\hat{\theta_1} = 2.5769$. Values of test statistics and achieved significance levels (p-values) obtained by simulated (in "real time") test statistic distributions under different values of N are given in Table 5.7.

Parameters estimated	S^*	$P\{S_n > S^*\}$				
		N = 1,000	N = 5,000	N = 10,000	N = 100,000	N = 1,000,000
K	0.59361	0.6380	0.6532	0.6535	0.6552	0.6556
CMS	0.05380	0.5390	0.5550	0.5555	0.5582	0.5582
AD	0.35021	0.5280	0.5498	0.5496	0.5485	0.5480

Table 5.7. *P-values of different tests under different volumes of simulation*

It should be noted that distributions of non-parametric goodness-of-fit test statistics [5.1–5.3] for $\hat{\theta_0} = 2.4706$ and $\hat{\theta_1} = 2.5769$ differ substantially from corresponding distributions under different combinations of integer values of θ_0 and θ_1.

5.4. Conclusions

In this chapter, implementation of a software has been discussed, which makes it possible to test composite hypotheses with the use of non-parametric goodness-of-fit tests in the cases when statistic distributions depend on specific values of the observed distributions.

The interactive mode is implemented for the Kolmogorov, the CMS and the AD tests, as well as the Kuiper test [KUI 60], the Watson test [WAT 61] and the test by Zhang *et al*. [ZHA 02, ZHA 05, ZHA 06].

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