Models of Statistic Distributions of Nonparametric Goodness-of-fit Tests in Composite Hypotheses Testing in Case of Double Exponential Law

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Abstract: In this paper are presented more precise results (tables of percentage points and statistic distribution models) for the nonparametric goodness-of-fit tests in testing composite hypotheses using the maximum likelihood estimate (MLE) for double exponential distribution law. Statistic distributions of the nonparametric goodness-of-fit tests are investigated by the methods of statistical simulation.
Keywords: goodness-of-fit test, composite hypotheses testing Kolmogorov test, Cramer-Mises-Smirnov test, Anderson-Darling test, double exponential distribution.

1 Introduction

In composite hypotheses testing of the form $H_0: F(x) \in \{F(x,\theta), \theta \in \Theta\}$, when the estimate $\hat{\theta}$ of the scalar or vector distribution parameter $F(x,\theta)$ is calculated by the same sample, the nonparametric goodness-of-fit Kolmogorov, ω^2 Cramer-Mises-Smirnov, Ω^2 Anderson-Darling tests lose the free distribution property.

The value

$$D_n = \sup_{|x|<\infty} \left| F_n(x) - F(x,\theta) \right| ,$$

where $F_n(x)$ is the empirical distribution function, n is the sample size, is used in Kolmogorov test as a distance between the empirical and theoretical laws. In testing hypotheses, a statistic with Bolshev correction (Bolshev, 1987) of the form (Bolshev and Smirnov, 1983)

$$S_K = \frac{6nD_n + 1}{6\sqrt{n}}, \tag{1}$$

where $D_n = \max(D_n^+, D_n^-)$,

$$D_n^+ = \max_{1 \le i \le n} \left\{ \frac{i}{n} - F(x_i,\theta) \right\}, \; D_n^- = \max_{1 \le i \le n} \left\{ F(x_i,\theta) - \frac{i-1}{n} \right\},$$

n is the sample size, $x_1, x_2, \ldots, x_n$ are sample values in increasing order is usually used. The distribution of statistic (1) in testing simple hypotheses obeys the Kolmogorov distribution law $K(S)$ (Bolshev and Smirnov, 1983).

In ω^2 Cramer-Mises-Smirnov test, one uses a statistic of the form

$$S_\omega = n\omega_n^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left\{ F(x_i,\theta) - \frac{2i-1}{2n} \right\}^2 , \tag{2}$$

and in test of Ω^2 Anderson-Darling type (Anderson and Darling, 1952, 1954), the statistic of the form

$$S_\Omega = -n - 2\sum_{i=1}^{n}\left\{\frac{2i-1}{2n}\ln F(x_i,\theta) + \left(1-\frac{2i-1}{2n}\right)\ln(1-F(x_i,\theta))\right\}. \quad (3)$$

In testing a simple hypothesis, statistic (2) obeys the distribution $a1(S)$ and statistic (3) obeys the distribution $a2(S)$ (see Bolshev and Smirnov, 1983).

In composite hypotheses testing, the conditional distribution law of the statistic $G(S|H_0)$ is affected by a number of factors: the form of the observed law $F(x,\theta)$ corresponding to the true hypothesis H_0; the type of the parameter estimated and the number of parameters to be estimated; sometimes, it is a specific value of the parameter (e.g., in the case of gamma-distribution and beta-distribution families); the method of parameter estimation. The distinctions in the limiting distributions of the same statistics in testing simple and composite hypotheses are so significant that we cannot neglect them.

The paper (Kac *et al.*, 1955) was a pioneer in investigating statistic distributions of the nonparametric goodness-of-fit tests with composite hypotheses. Then, for the solution to this problem, various approaches where used (Darling, 1955, 1957]), (Durbin, 1973, 1975), (Martinov,1978), (Pearson and Hartley, 1972), (Stephens, 1970, 1974), (Chandra *et al.*, 1981), (Tyurin, 1984), (Dzhaparidze and Nikulin, 1982), (Nikulin, 1992).

In our research (Lemeshko and Postovalov, 1998, 2001a, 2001b, 2002), (Lemeshko and Maklakov, 2004), (Lemeshko, 2004), statistic distributions of the nonparametric goodness-of-fit tests are investigated by the methods of statistical simulating, and for constructed empirical distributions approximate models of law are found. The results obtained were used to develop recommendations for standardization (R 50.1.037-2002, 2002).

2 Statistic distributions of the tests in testing composite hypotheses concerning double exponential law

In testing composite hypotheses for distribution law with density

$$f(x,\boldsymbol{\theta}) = \frac{\theta_0}{2\theta_1\Gamma(1/\theta_0)}\exp\left\{-\left(\frac{|x-\theta_2|}{\theta_1}\right)^{\theta_0}\right\}, \quad (4)$$

distributions $G(S|H_0)$ of non-parametric goodness-of-fit tests statistics depend on specific value of shape parameter θ_0.

The family (4) defines a set of symmetric laws, special cases of which are normal distribution ($\theta_0 = 2$) and Laplace distribution ($\theta_0 = 1$). Sometimes this distribution is called double-sided exponential, although usually $\theta_0 = 1$ is implied.

The feature in behavior of non-parametric goodness-of-fit tests statistics $G(S|H_0)$ S_ω when testing composite hypotheses for family (4) is that with shape parameter growing up to $\theta_0 \approx 1.64$, distributions $G(S|H_0)$ are shifting to the right, and with the following growth, the shift starts in the opposite direction (see Fig.1).

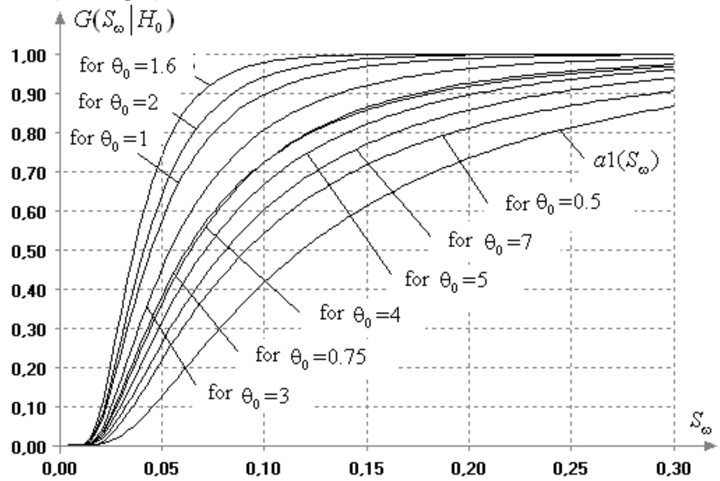

Fig. 1. Statistic distributions (2) of Cramer-Mises-Smirnov goodness-of-fit tests in composite hypotheses testing concerning family (4) if MLE is used for all three parameters subject to value θ_0

Upper percentage points and the models of limiting statistic distributions of Kolmogorov, Cramer-von Mises-Smirnov and Anderson-Darling tests have been constructed for the values of shape parameter θ_0=0.5, 0.75, 1, 1.6, 2, 3, 4, 5, 7 when MLEs were used. The results for the shape parameter θ_0=1.6 are presented as an example in table 1.These results define more accurately supplemented results that were presented in (Lemeshko *et al.*, 2004). If the value of form parameter θ_0 is not congruent with tabular, interpolation could be used to obtain approximate percentage points.

Distributions $G(S|H_0)$ of the Kolmogorov, Cramer-Mises-Smirnov and the Anderson-Darling statistics are best approximated by the family of the III type beta-distributions with the density function

$$B_3(\theta_0,\theta_1,\theta_2,\theta_3,\theta_4) = \frac{\theta_2^{\theta_0}}{\theta_3 \mathrm{B}(\theta_0,\theta_1)} \frac{\left(\frac{x-\theta_4}{\theta_3}\right)^{\theta_0-1}\left(1-\frac{x-\theta_4}{\theta_3}\right)^{\theta_1-1}}{\left[1+(\theta_2-1)\frac{x-\theta_4}{\theta_3}\right]^{\theta_0+\theta_1}},$$

by gamma-distributions family $\gamma(\theta_0,\theta_1,\theta_2) = \frac{1}{\theta_1^{\theta_0}\Gamma(\theta_0)} (x-\theta_2)^{\theta_0-1} e^{-(x-\theta_2)/\theta_1}$,

by the family of the *Sb*-Johnson distributions

$$Sb(\theta_0,\theta_1,\theta_2,\theta_3) = \frac{\theta_1\theta_2}{(x-\theta_3)(\theta_2+\theta_3-x)} \exp\left\{-\frac{1}{2}\left[\theta_0 - \theta_1 \ln\frac{x-\theta_3}{\theta_2+\theta_3-x}\right]^2\right\}$$

or by the family of the *Sl*-Johnson distributions

$$Sl(\theta_0,\theta_1,\theta_2,\theta_3)=\frac{\theta_1}{x-\theta_3\ \sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left[\theta_0+\theta_1\ln\frac{x-\theta_3}{\theta_2}\right]^2\right\}.$$

The tables of percentage points and statistic distributions models were constructed by modeled statistic samples with the size $N=10^6$ (N is the number of runs in simulation). This number ensures the deviation of the empirical p.d.f. $G_N(S|H_0)$ from the theoretical (true) to be less than 10^{-3}. In this case, the samples of pseudorandom variables, belonging to $F(x,\theta)$, were generated with the size $n=10^3$. For such value of n statistic p.d.f. $G(S_n|H_0)$ almost coincides with the limit p.d.f. $G(S|H_0)$.

Table 1. Upper percentage points and models of limiting statistic distributions of the nonparametric goodness-of-fit test when MLE are used (for θ_0 =1.6)

Parameter	Percentage points			Model
	0.9	0.95	0.99	
for Kolmogorov's test				
θ_0	1.216	1.351	1.621	B_3(4.2366, 5.7254, 2.8969, 2.4200, 0.330)
θ_1	1.185	1.322	1.596	B_3(4.3698, 5.2853, 3.3545, 2.3863, 0.318)
θ_2	0.851	0.923	1.069	B_3(5.4129, 7.6381, 2.1289, 1.3936, 0.290)
θ_0, θ_1	1.141	1.280	1.557	B_3(4.9730, 4.5743, 4.6422, 2.3576, 0.29)
θ_0, θ_2	0.828	0.898	1.039	B_3(6.2506, 7.4916, 2.5914,1.4130, 0.275)
θ_1, θ_2	0.770	0.831	0.953	B_3(5.3623, 7.3149, 2.1379, 1.1702, 0.29))
θ_0, θ_1, θ_2	0.704	0.759	0.873	B_3(7.4853, 7.2752, 3.2095, 1.14609, 0.260)
for Cramer-Mises-Smirnov's test				
θ_0	0.339	0.453	0.735	*Sb*(3.6139, 1.0337, 3.400, 0.013)
θ_1	0.325	0.440	0.723	*Sb*(2.7348, 0.9148, 1.800, 0.016)
θ_2	0.121	0.149	0.219	B_3(4.5239, 3.7332, 15.6889, 0.6596, 0.009)
θ_0, θ_1	0.314	0.429	0.711	*Sb*(2.3111, 0.8115, 1.350, 0.016)
θ_0, θ_2	0.109	0.134	0.194	B_3(4.2190, 3.9949, 12.6139, 0.5642, 0.0087)
θ_1, θ_2	0.087	0.104	0.143	B_3(4.5491, 4.8658, 9.0448, 0.4000, 0.008)
θ_0, θ_1, θ_2	0.069	0.083	0.118	B_3(6.8750, 4.6392, 18.020, 0.3937, 0.006)
for Anderson-Darling's test				
θ_0	1.819	2.383	3.774	B_3(3.7982, 2.4042, 26.2612, 10.00, 0.095)
θ_1	1.735	2.304	3.697	B_3(3.6908, 2.1990, 32.1310, 10.00, 0.10)
θ_2	0.864	1.052	1.513	B_3(4.0782, 5.1594, 17.0570, 7.900, 0.09)
θ_0, θ_1	1.669	2.235	3.630	B_3(4.6625, 1.4267, 33.5120, 4.500, 0.09)
θ_0, θ_2	0.716	0.863	1.207	B_3(4.5576, 4.2326, 10.9573, 3.23142, 0.08)
θ_1, θ_2	0.589	0.695	0.941	B_3(4.5825, 5.3012, 7.9243, 2.5555, 0.0775)
θ_0, θ_1, θ_2	0.492	0.587	0.819	B_3(5.08840, 5.2459, 10.6760, 2.4738, 0.068)

3 Conclusions

In this work are presented more precise models of the statistic distributions of the nonparametric goodness-of-fit tests for testing composite hypotheses with the distributions family (4).

It should be stressed, that obtained percentage points and models guarantee proper implementation of the nonparametric goodness-of-fit tests in statistic analysis problems if MLE is used. These results can't be used with other estimations because statistic distributions of these tests are essential depend on estimation method (Lemeshko *et al*., 2001b).

In the case of the I, II, III type beta-distribution families' statistic distributions depend on a specific value of two form parameter of these distributions. Statistic distributions models and tables of percentage points for various combinations of values of two form parameters (more than 1500 models) were constructed in the thesis of Lemeshko S.B. and partly were published in the paper (Lemeshko *et al*., 2007).

Note that, in composite hypotheses testing, power of the nonparametric goodness-of-fit tests, generally, essentially higher (if MLE is used), than in simple hypotheses testing.

The results of comparative analysis of goodness-of-fit tests power (nonparametric and χ^2 type) subject to some sufficiently close pair of alternative are presented in (Lemeshko *et al*., 2007), and are in more detail stated in (Lemeshko *et al*., 2008a, 2008b).

The authors hope that release of the article will be conductive to decrease mistake amount, committed in statistic analysis problems if nonparametric goodness-of-fit tests are used (Lemeshko, 2004).

This research was supported by Federal Education Agency of Russian Federation Ministry of Education in the Analytical departmental purposeful program framework "Development of Higher School Potential", (project № 2.1.2/3970).

References

Anderson T.W., and Darling D.A. 1952. *Asymptotic theory of certain "goodness of fit" criteria based on stochastic processes*. Ann.Math. Statist., 23: 193-212.

Anderson T.W., and Darling D.A. 1954. *A test of goodness of fit*. J. Amer. Stist. Assoc., 29: 765-769.

Bolshev L.N. 1987. *On the question on testing some composite statistical hypotheses*. In *Theory of Probability and Mthematical Statistics*. Selected Works. Nauka, Moscow, 5-63.

Bolshev L.N., Smirnov N.V. 1983. *Tables of Mathematical Statistics*. Moscow: Science. (in Russian)

Chandra M., Singpurwalla N.D., Stephens M.A. 1981. *Statistics for Test of Fit for the Extrem–Value and Weibull Distribution*. J. Am. Statist. Assoc. 76(375): 729-731.

Darling D.A. 1955. *The Cramer-Smirnov test in the parametric case*. Ann. Math. Statist., 26: 1-20.

Darling D.A. 1957. *The Cramer-Smirnov test in the parametric case*. Ann. Math. Statist., 28: 823-838.

Durbin J. 1973. *Weak convergence of the sample distribution function when parameters are estimated*. Ann. Statist., 1: 279-290.

Durbin J. 1975. *Kolmogorov-Smirnov tests when parameters are estimated with applications to tests of exponentiality and tests of spacings*. Biometrika, 62: 5-22.

Dzhaparidze K.O., and Nikulin M.S. 1982. *Probability distribution of the Kolmogorov and omega-square statistics for continuous distributions with shift and scale parameters*. J. Soviet Math., 20: 2147-2163.

Kac M., Kiefer J., Wolfowitz J. 1955. *On Tests of Normality and Other Tests of Goodness of Fit Based on Distance Methods*. Ann. Math. Stat. 26: 189-211.

Lemeshko B.Yu. 2004. *Errors when using nonparametric fitting criteria*. Measurement Techniques, 47(2): 134-142.

Lemeshko B.Yu., Postovalov S.N. 1998. *Statistical Distributions of Nonparametric Goodness-of-Fit Tests as Estimated by the Sample Parameters of Experimentally Observed Laws*. Industrial laboratory, 64(3): 197-208.

Lemeshko B.Yu., Postovalov S.N. (2001a). Application of the Nonparametric Goodness-of-Fit Tests in Testing Composite Hypotheses. *Optoelectronics, Instrumentation and Data Processing* 37(2): 76-88.

Lemeshko B.Yu., Postovalov S.N. 2001b. *About dependence of statistics distributions of the nonparametric tests and their power from a method of the parameters estimation*. Industrial laboratory. Diagnostics of materials. 2001. 67(7): 62-71.

Lemeshko B.Yu., Postovalov S.N. 2002. *The nonparametric goodness-of-fit tests about fit with Johnson distributions in testing composite hypotheses*. News of the SB AS HS 1(5): 65-74. (in Russian)

Lemeshko B.Yu., Maklakov A.A. 2004. *Nonparametric Test in Testing Composite Hypotheses on Goodness of Fit Exponential Family Distributions*. Optoelectronics, Instrumentation and Data Processing, 40(3): 3-18.

Lemeshko B.Yu., Lemeshko S.B., Postovalov S.N. 2007. The power of goodness-of-fit tests for close alternatives. *Measurement Techniques* 50(2): 132-141.

Lemeshko B.Yu., Lemeshko S.B., Postovalov S.N. 2008a. *The comparative analysis of the power of goodness-of-fit tests at close competing hypotheses. I. Tests of simple hypotheses*. Sib. Zh. Ind. Mat., **11**(2): 96–111. (in Russian)

Lemeshko B.Yu., Lemeshko S.B., Postovalov S.N. 2008b. *The comparative analysis of the power of goodness-of-fit tests at close competing hypotheses. II. Tests of composite hypotheses*. Sib. Zh. Ind. Mat., **11**(4): 78–93. (in Russian)

Lemeshko S.B., Lemeshko B.Yu. 2007. *Statistic distributions of the nonparametric goodness-of-fit tests in testing hypotheses relative to beta-distributions*. News of the SB AS HS 2(9): 6-16. (in Russian)

Martinov G.V. 1978. *Omega-Quadrate Tests*. Moscow: Science. (in Russian)

Nikulin M.S. 1992. *A variant of the generalized omega-square statistic*. J. Soviet Math., 61 (4):1896-1900.

Pearson E.S., Hartley H.O. 1972. *Biometrica Tables for Statistics*. Vol. 2. Cambridge: University Press.

R 50.1.037-2002. 2002. *Recommendations for Standardization. Applied statistics. Rules of check of experimental and theoretical distribution of the consent. Part II. Nonparametric goodness–of–fit test.* Moscow: Publishing house of the standards. (in Russian)

Stephens M.A. 1970. *Use of Kolmogorov–Smirnov, Cramer – von Mises and Related Statistics – Without Extensive Table*. J. R. Stat. Soc., 32: 115-122.

Stephens M.A. 1974. *EDF Statistics for Goodness of Fit and Some Comparisons*. J. Am. Statist. Assoc., 69: 730-737.

Tyurin Yu.N. 1984. *On the Limiting Kolmogorov-Smirnov Statistic Distribution for Composite Hypothesis*. News of the AS USSR. Ser. Math., 48(6): 1314-1343.