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BARTLETT AND COCHRAN TESTS IN MEASUREMENTS WITH PROBABILITY LAWS DIFFERENT FROM NORMAL

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Statistical simulation methods have been applied to the distributions of classical statistics used in testing hypotheses on variance in a series of samples. The statistics used in the Bartlett and Cochran tests have been derived as tables of percentage points, which are used in observed laws describing an exponential family of distributions.

Key words: *hypothesis testing, mathematical expectation, variance, percentage points, Bartlett and Cochran tests.*

In statistical quality control, one usually checks for perturbations on tests that check hypotheses on the constancy of the variances in a control parameter or equality of that parameter to some nominal value. Analogous problems in hypothesis testing arise in measurements. Published sources [1, 2] and standards [3] deal with the use of Bartlett [4] and Cochran [5] tests, in hypothesis testing on equality of the variances of a set of samples. In [3], Cochran's test is envisaged for identifying excursions in physicochemical measurements.

The following is the hypothesis tested on the constancy of the variance in m sets:

$$H_0 : \sigma_1^2 = \sigma_2^2 = ... = \sigma_m^2,$$

which a competing hypothesis is

$$H_1 : \sigma_{i_1}^2 \neq \sigma_{i_2}^2 ,$$

where the inequality is obeyed at least for one pair of subscripts i_1, i_2.

For example, in monitoring some parameter, hypothesis H_1 leads one to assert that at least for two instants in sampling from a general set of sampling instants m (m samples taken at different times), the variance has different values.

The basic assumption in constructing the Bartlett and Cochran tests and deriving the limiting distributions of the statistics for them is that a normal distribution applies for the observed random quantities (measurement errors).

The errors of measuring instruments far from always are described by normal distributions [6]; it is clear also that in quality control, deviations in the monitored parameter from the nominal (given) value when the process is stationary do not always obey a normal law. The process may satisfy the conditions imposed, e.g., the mathematical expectation coincides with the nominal value of the parameter, while the variance does not exceed a given value.

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How do these tests behave when the errors of the measurements are not normally distributed? Is it possible to use these tests in their classical form in a varying situation or what actions lead to incorrect results?

We examined the distributions of the statistics for these tests with various degrees of deviation (measurement errors) from normal distribution and thus devised recommendations on using those tests under such conditions. These results supplement [7] on the behavior of the statistics used in testing hypotheses on the variances and mathematical expectations. As in [7], we used a method of statistical simulation and computer analysis that has been found to be sound in studying statistical regularities in [8, 9].

Bartlett test. The statistic for the Bartlett test is given by [2]:

$$\chi^2 = M\left[1 + \frac{1}{3(m-1)}\left(\sum_{i=1}^{m}\frac{1}{\nu_i} - \frac{1}{N}\right)\right]^{-1}, \tag{1}$$

where $\nu_i = n_i$ if the mathematical expectation is known or $\nu_i = n_i - 1$ if it is unknown (n_i are the sample volumes); $N = \sum_{i=1}^{m}\nu_i$;

$$M = N\ln\left(\frac{1}{N}\sum_{i=1}^{m}\nu_i S_i^2\right) - \sum_{i=1}^{m}\nu_i \ln S_i^2,$$

where S_i^2 are the estimators for the sample variances.

If the mathematical expectation is unknown, the estimators are

$$S_i^2 = \frac{1}{n_i - 1}\sum_{j=1}^{n_i}(X_{ji} - \bar{X}_i)^2,$$

where $\bar{X}_i = \frac{1}{n_i}\sum_{j=1}^{n_i}X_{ji}$, $\nu_i = n_i - 1$; if hypothesis H_0 is true, all the $\nu_i > 3$ and the samples are extracted from a normal population, then the (1) statistic approximately obeys a χ^2_{m-1} distribution.

If the measurements are normally distributed, the distribution for the (1) statistic is almost independent of the sample volume. Figure 1 shows virtually coincident distributions for the Bartlett statistic of (1) with various sample volumes (n = 10, 50, and 100). This means that if the measurements fit a normal distribution, the conclusions remain correct even for very small volumes in the samples.

On the other hand, the distribution of the (1) statistic is very sensitive to deviations in the observed law from normal. The form of the (1) statistic distribution has been examined for various observed laws, in particular for the case where the samples fit a logit law with density

$$f(x) = \frac{\pi}{\theta_2\sqrt{3}}\exp\left\{-\frac{\pi(x-\theta_1)}{\theta_2\sqrt{3}}\right\}\Bigg/\left[1 + \exp\left\{-\frac{\pi(x-\theta_1)}{\theta_2\sqrt{3}}\right\}\right]^2,$$

a Laplace law with density

$$f(x) = \frac{1}{2\theta_2}\exp\left\{-\frac{|x-\theta_1|}{\theta_2}\right\},$$

and an exponential family of distributions with various shape parameters and density

$$\mathrm{De}(\theta_3) = f(x) = \frac{\theta_3}{2\theta_2\Gamma(1/\theta_3)}\exp\left\{-\left(\frac{|x-\theta_1|}{\theta_2}\right)^{\theta_3}\right\}, \tag{2}$$

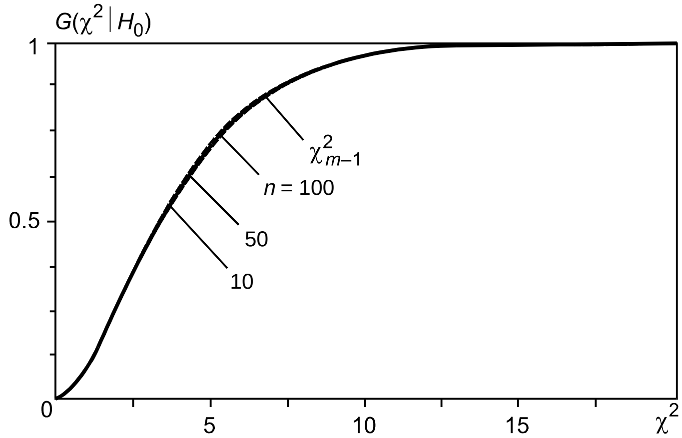

Fig. 1. Distributions for the classical Bartlett statistic for various sample volumes and m = 5.

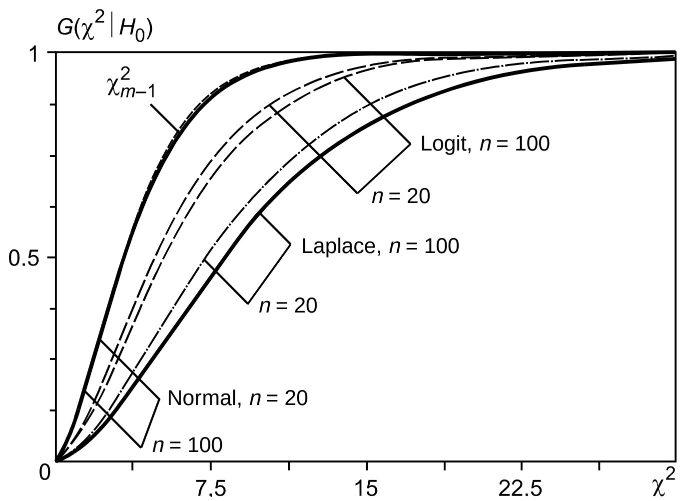

Fig. 2. Distributions for the Bartlett statistic when the observed parameter differs from a normal distribution for various sample volumes and m = 5.

where θ_3 is the shape parameter. The normal and Laplace distributions are particular cases of this family with values for the shape parameter of 2 and 1 respectively. Family (2) may be a good model for the error distributions for various measuring systems.

Figure 2 shows how the distributions for statistic (1) are dependent on the form of the observed law for various sample volumes. When the distribution of the observed parameter deviates from a normal law, distribution for the (1) Bartlett statistic differs substantially from a χ^2_{m-1} distribution. The distribution of the statistic becomes more dependent on the sample volume than is the case for the normal law.

Consequently, if one operates with a classical limiting χ^2_{m-1} distribution when the observed quantity fits a Laplace distribution or a logit one, there is a high probability of rejecting the hypothesis H_0 on equality of the variances even when the hypothesis is correct. Figure 3 shows how the distribution of the Bartlett statistic alters if the measurement results belong to the exponential family with various values of the shape parameter. The shapes show that when the measurement results belong to an exponential family more flat-topped than a normal one, then if one uses the classical limiting distribution, there is a high probability of taking the hypothesis H_0 as incorrect.

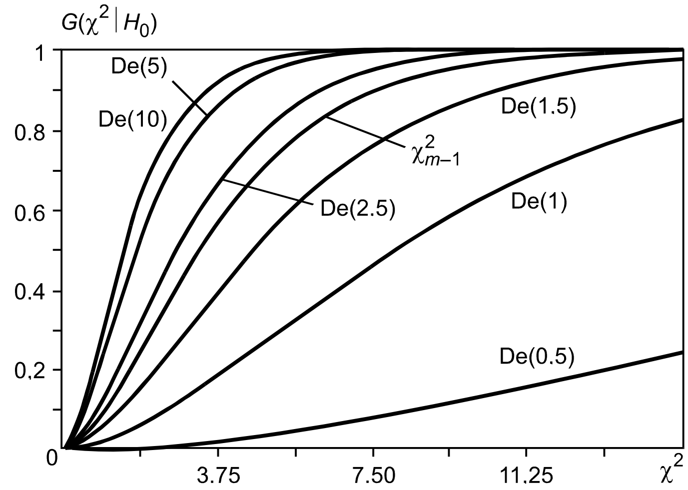

Fig. 3. Distributions of the Bartlett statistic for distributions in the exponential family De(θ_3) with various shape parameters for n = 100 and m = 5.

The Fig. 3 picture illustrates well the arguments [10] presented in [11] on the use of the classical Bartlett test [4] when the assumptions about normality are violated. If the distribution of the observed quantities is more flat-topped than a normal one (with negative values for the coefficient of excess

$$\gamma_2 = \mu_4/\sigma^4 - 3 < 0,$$

with $\mu_4 = E[X - \mu)^4]$ the fourth central moment and σ^2 the variance), the classical Bartlett test [4] conceals the difference in the variances, while if it is more sharp-peaked (with $\gamma_2 > 0$), it finds a difference in the variances that is not present (Figure 6 shows the analogous picture for the Cochran test).

For distributions deviating from normal, the distributions of the Bartlett statistic are dependent substantially on the sample volume n, but they converge quite well to certain limiting laws.

If the observed quantities belong to an exponential family of distributions, we have used Monte Carlo simulation to construct tables for the upper percentage points (1, 5, and 10%) of the Bartlett statistic for shape parameters θ_3 = 0.5, 1, 3, 5, and 10 for various m and n. The percentage points were constructed from the simulated empirical distribution for the statistics of volume 50000 with averaging over a series of experiments. In the case of the Laplace distribution (θ_3 = 1), Table 1 gives the percentage points obtained.

Cochran Test. Here, when all the n_i are identical ($n_1 = n_2 = ... = n_m = n$), one can use the simpler Cochran test. The statistic Q for the Cochran test is [2]

$$Q = S^2_{\max}/(S^2_1 + S^2_2 + ... + S^2_m), \tag{3}$$

where $S^2_{\max} = \max(S^2_1, S^2_2, ..., S^2_m)$ and m is the number of independent variance estimators (number of samples).

The distribution of the Cochran statistic is very much dependent on the sample volume, so the reference literature gives only tables of the percentage points [2], which are used in hypothesis testing. Fig. 4 shows computer simulation results for the distribution of the (3) statistic with various sample volumes. In this case, the number of variance estimators is m = 5.

The Cochran test resembles the Bartlett one in being used on the assumption that the measurements fit a normal distribution, so interest attaches to the changes in the distribution for the Cochran statistic of (3) with certain deviations of the measurements (control parameter) from a normal distribution. Figure 5 shows the distribution of the (3) statistic when H_0 is correct for the quality of the variances and the samples belong to various laws: normal, logit, and Laplace. The differences in the (3) statistic distributions persist for any sample volumes.

TABLE 1. Upper Percentage Points ($\alpha \cdot 100\%$) for the Bartlett Statistic Constructed from *m* Independent Variance Estimates, Each of Which with $\nu = n - 1$ Degrees of Freedom. The Sample Belongs to the Distribution in the Exponential Family with Shape Parameter $\theta_3 = 1$ (Laplace distribution)

m	*n*	α				
		0.15	0.1	0.05	0.02	0.01
2	10	3.96	5.10	7.04	9.57	11.37
	50	4.78	6.21	8.78	12.29	14.93
	100	4.92	6.42	9.07	12.70	15.43
3	10	7.25	8.71	11.16	14.27	16.51
	50	8.78	10.63	13.74	17.79	20.80
	100	8.95	10.83	14.06	18.29	21.40
4	10	10.29	11.98	14.78	18.24	20.82
	50	12.26	14.40	17.91	22.42	25.80
	100	12.63	14.83	18.52	23.30	26.87
5	10	13.04	14.96	18.09	21.93	24.81
	50	15.65	18.02	21.93	26.83	30.54
	100	16.05	18.47	22.48	27.69	31.54
6	10	15.75	17.85	21.23	25.43	28.49
	50	18.75	21.31	25.51	30.76	34.63
	100	19.22	21.87	26.23	31.45	35.43
7	10	18.35	20.60	24.19	28.66	31.98
	50	21.87	24.63	29.06	34.63	38.71
	100	22.45	25.30	29.87	35.60	39.87
8	10	21.00	23.37	27.22	32.08	35.66
	50	24.88	27.79	32.50	38.26	42.49
	100	25.59	28.58	33.41	39.50	43.77
9	10	23.53	26.06	30.05	35.10	38.65
	50	27.88	30.96	35.90	42.09	46.44
	100	28.53	31.71	36.72	43.11	47.48
10	10	26.04	28.68	32.99	38.18	41.85
	50	30.78	33.98	39.20	45.36	49.88
	100	31.55	34.82	40.14	46.72	51.32

Figure 6 shows that there are substantial changes in the Cochran statistic distribution if the measurements fit a distribution in the exponential family for the various shape parameters. The form of the (3) statistic distribution (Figs. 5 and 6) indicates that if the measurement results violate the assumption of normality, but this is not considered, then the hypothesis on equality of the variances may be rejected with a high probability although it is correct and may be adopted in the case when it is incorrect.

It has been asserted [2] that the Cochran test is somewhat inferior in power to the Bartlett one. The distributions for the Cochran statistic are very much dependent on the sample volume even for a normal distribution and are very much dependent on the form of the observed law. It might seem that this makes it unsuitable for arbitrary observed laws, but in fact our studies have shown that the Cochran test is more powerful than the Bartlett one if the observations fit a normal distribution.

For example, Table 2 gives values for the power $1 - \beta$ of these tests relative to three distinct alternatives with various sample volumes *n* for probabilities of errors of the first kind $\alpha = 0.1$, 0.05, and 0.01. The competing hypothesis suppos-

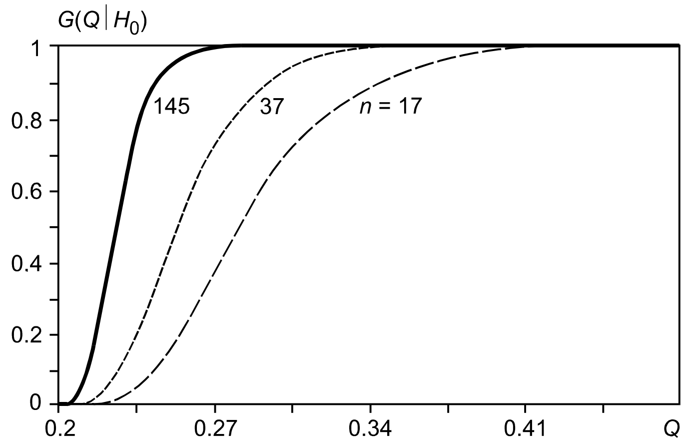

Fig. 4. Distributions for the Cochran statistic with various sample volumes with $m = 5$.

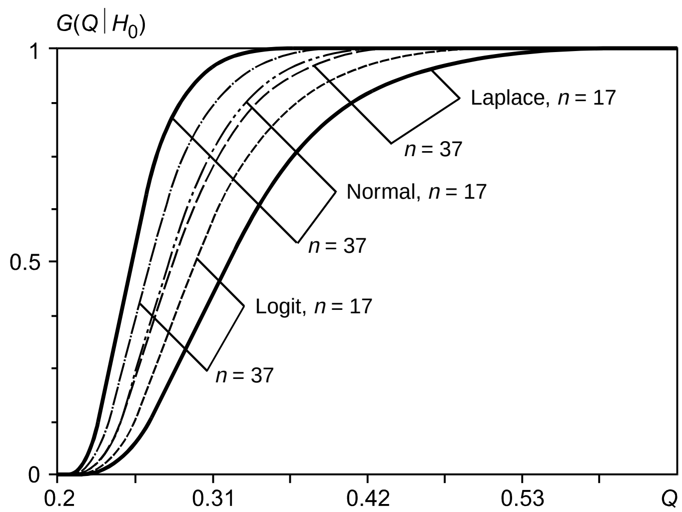

Fig. 5. Distributions for the Cochran statistic for deviations in the distribution of the observed parameter from normal with various sample volumes and $m = 5$.

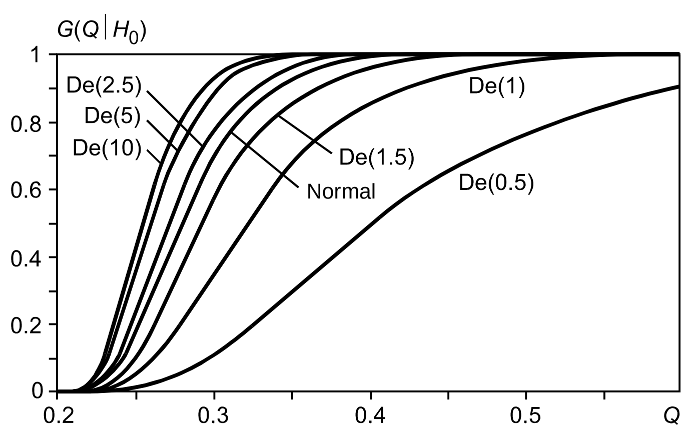

Fig. 6. Distributions for the Cochran statistic in the case of distributions in the exponential family with various values for the shape parameter with $n = 17$ and $m = 5$.

TABLE 2. Power $1 - \beta$ of the Bartlett and Cochran Tests with Respect to the Alternatives H_1^1, H_1^2, H_1^3

n	α	$H_1^1: \sigma_m = 1.05\sigma_0$ for tests		$H_1^2: \sigma_m = 1.1\sigma_0$ for tests		$H_1^3: \sigma_m = 1.2\sigma_0$ for tests	
		Bartlett	Cochran	Bartlett	Cochran	Bartlett	Cochran
200	0.1	0.1706	0.2342	0.3600	0.4778	0.8346	0.9196
	0.05	0.1030	0.1078	0.2534	0.3022	0.7568	0.8370
	0.01	0.0274	0.0306	0.1064	0.1362	0.5558	0.6708
500	0.1	0.2608	0.3488	0.6712	0.7976	0.9968	0.9990
	0.05	0.1682	0.1938	0.5554	0.6598	0.9926	0.9974
	0.01	0.0608	0.0726	0.3330	0.4288	0.9702	0.9860
1000	0.1	0.4556	0.4990	0.9432	0.9686	0.9998	1
	0.05	0.3340	0.3816	0.8998	0.9410	0.9998	1
	0.01	0.1368	0.2034	0.7422	0.8560	0.9996	1

TABLE 3. Upper Percentage Points (1%) for the Cochran Statistic Constructed from m Independent Variance Estimators Each of Which Has ν Degrees of Freedom. The Samples Belong to a Distribution of the Exponential Family with Shape Parameter $\theta_3 = 1$ (Laplace distribution)

m	ν												
	1	2	3	4	5	6	7	8	9	10	16	36	144
2	–	0.997	0.990	0.978	0.975	0.965	0.949	0.939	0.926	0.917	0.872	0.786	0.659
3	0.997	0.970	0.941	0.908	0.886	0.866	0.841	0.827	0.808	0.791	0.724	0.606	0.475
4	0.981	0.931	0.881	0.855	0.800	0.787	0.744	0.734	0.704	0.710	0.612	0.499	0.372
5	0.953	0.888	0.829	0.775	0.735	0.700	0.672	0.655	0.635	0.622	0.538	0.415	0.307
6	0.936	0.836	0.765	0.730	0.676	0.638	0.615	0.593	0.576	0.552	0.470	0.363	0.259
7	0.897	0.796	0.727	0.678	0.619	0.592	0.578	0.550	0.521	0.505	0.420	0.325	0.228
8	0.874	0.772	0.683	0.624	0.577	0.555	0.520	0.499	0.471	0.457	0.380	0.287	0.202
9	0.846	0.731	0.654	0.596	0.550	0.510	0.485	0.453	0.440	0.424	0.342	0.263	0.180
10	0.824	0.715	0.609	0.553	0.511	0.479	0.450	0.425	0.406	0.394	0.322	0.242	0.164
12	0.774	0.648	0.565	0.501	0.458	0.427	0.396	0.377	0.361	0.347	0.282	0.209	0.138
15	0.716	0.577	0.496	0.450	0.394	0.366	0.340	0.328	0.304	0.297	0.242	0.176	0.113
20	0.624	0.501	0.408	0.355	0.327	0.303	0.279	0.269	0.251	0.239	0.190	0.134	0.086
24	0.572	0.441	0.369	0.320	0.295	0.263	0.245	0.230	0.218	0.207	0.164	0.115	0.072
30	0.512	0.383	0.315	0.274	0.257	0.218	0.206	0.197	0.184	0.173	0.137	0.094	0.059
40	0.428	0.321	0.258	0.221	0.200	0.184	0.163	0.154	0.143	0.137	0.106	0.073	0.045
60	0.331	0.246	0.194	0.167	0.144	0.128	0.118	0.106	0.103	0.097	0.074	0.050	0.031

es that one of the sets, e.g., the set with number m, has a somewhat different variance. We considered the alternatives $H_1^1: \sigma_m = 1.05\sigma_0$; $H_1^2: \sigma_m = 1.1\sigma_0$; and $H_1^3: \sigma_m = 1.2\sigma_0$; the other $m - 1$ samples belonged to a normal distribution having $\sigma = \sigma_0$ ($H_0: \sigma_1^2 = \sigma_2^2 = ... = \sigma_m^2 = \sigma_0^2$). The powers in Table 2 are given for $m = 5$ and sufficiently large sample volumes, since for small n the power of these tests is very low, i.e., the capacity to distinguish similar alternatives.

The [3] proposes using the Cochran test for small sample volumes $n = 2$–6, so it must be borne in mind that for such values of n, one can distinguish fairly reliably only fairly remote alternatives, where the variances differ by large factors.

TABLE 4. Upper Percentage Points (5%) for the Cochran Statistic Constructed from *m* Independent Variance Estimators Each of Which Has ν Degrees of Freedom. The Samples Belong to a Distribution of the Exponential Family with Shape Parameter $\theta_3 = 1$ (Laplace distribution)

m	ν												
	1	2	3	4	5	6	7	8	9	10	16	36	144
2	0.999	0.984	0.966	0.947	0.928	0.910	0.894	0.885	0.870	0.862	0.815	0.728	0.623
3	0.978	0.924	0.879	0.842	0.804	0.784	0.760	0.740	0.725	0.710	0.641	0.548	0.443
4	0.938	0.852	0.803	0.747	0.706	0.679	0.658	0.635	0.623	0.604	0.534	0.447	0.346
5	0.894	0.788	0.717	0.683	0.647	0.606	0.585	0.557	0.547	0.524	0.466	0.377	0.284
6	0.857	0.737	0.662	0.616	0.570	0.551	0.526	0.495	0.482	0.464	0.407	0.327	0.239
7	0.811	0.690	0.620	0.570	0.530	0.500	0.481	0.454	0.435	0.421	0.364	0.283	0.210
8	0.765	0.650	0.571	0.529	0.492	0.460	0.433	0.416	0.398	0.389	0.333	0.256	0.185
9	0.736	0.609	0.547	0.501	0.452	0.431	0.407	0.390	0.371	0.357	0.303	0.233	0.167
10	0.710	0.578	0.511	0.465	0.426	0.398	0.379	0.359	0.344	0.330	0.276	0.214	0.152
12	0.652	0.531	0.461	0.415	0.379	0.353	0.333	0.319	0.306	0.296	0.244	0.182	0.128
15	0.590	0.466	0.406	0.357	0.329	0.306	0.285	0.276	0.262	0.246	0.204	0.151	0.104
20	0.508	0.394	0.340	0.297	0.269	0.251	0.234	0.220	0.208	0.200	0.163	0.117	0.080
24	0.461	0.348	0.304	0.264	0.241	0.222	0.207	0.191	0.182	0.172	0.139	0.100	0.067
30	0.401	0.307	0.258	0.224	0.202	0.187	0.170	0.163	0.153	0.146	0.115	0.082	0.055
40	0.338	0.263	0.212	0.182	0.167	0.148	0.139	0.130	0.120	0.106	0.083	0.064	0.042
60	0.262	0.195	0.159	0.137	0.120	0.109	0.100	0.093	0.087	0.084	0.064	0.045	0.029

TABLE 5. Upper Percentage Points (10%) for the Cochran Statistic Constructed from *m* Independent Variance Estimators Each of Which Has ν Degrees of Freedom. The Samples Belong to a Distribution of the Exponential Family with Shape Parameter $\theta_3 = 1$ (Laplace distribution)

m	ν												
	1	2	3	4	5	6	7	8	9	10	16	36	144
2	0.996	0.969	0.942	0.917	0.894	0.874	0.857	0.849	0.832	0.823	0.776	0.698	0.606
3	0.955	0.879	0.826	0.788	0.756	0.730	0.711	0.688	0.672	0.661	0.597	0.519	0.428
4	0.895	0.796	0.740	0.691	0.653	0.624	0.605	0.581	0.572	0.555	0.496	0.420	0.332
5	0.842	0.720	0.657	0.615	0.586	0.552	0.534	0.511	0.499	0.483	0.428	0.351	0.273
6	0.790	0.671	0.604	0.554	0.518	0.495	0.479	0.448	0.439	0.424	0.375	0.305	0.231
7	0.743	0.621	0.559	0.509	0.474	0.455	0.432	0.410	0.394	0.381	0.334	0.267	0.202
8	0.698	0.575	0.517	0.472	0.440	0.415	0.389	0.374	0.364	0.355	0.302	0.238	0.178
9	0.669	0.547	0.485	0.441	0.407	0.384	0.368	0.352	0.333	0.320	0.278	0.217	0.160
10	0.637	0.522	0.457	0.412	0.379	0.358	0.341	0.325	0.311	0.298	0.255	0.200	0.146
12	0.585	0.468	0.408	0.367	0.343	0.316	0.299	0.286	0.274	0.265	0.222	0.170	0.123
15	0.522	0.412	0.355	0.321	0.295	0.274	0.255	0.244	0.237	0.220	0.186	0.141	0.100
20	0.446	0.347	0.298	0.265	0.237	0.226	0.209	0.199	0.187	0.180	0.148	0.110	0.077
24	0.403	0.309	0.266	0.232	0.214	0.195	0.184	0.171	0.163	0.156	0.128	0.094	0.065
30	0.351	0.270	0.227	0.200	0.181	0.167	0.152	0.147	0.139	0.130	0.106	0.077	0.053
40	0.297	0.226	0.186	0.162	0.148	0.132	0.123	0.118	0.109	0.117	0.091	0.060	0.040
60	0.230	0.173	0.140	0.121	0.107	0.096	0.090	0.085	0.079	0.076	0.059	0.042	0.028

If the observed quantities fit distributions in the exponential family, tables can be constructed for the upper percentage points (1, 5, and 10%) of the Cochran statistic for shape parameters $\theta_3 = 0.5$, 1, 3, 5, and 10 with various m and n. The percentage points have been constructed from Monte Carlo simulation and were derived from the simulated experimental distributions of the statistics with volumes of 50000 and averaging over a series of experiments. In the case of the Laplace distribution ($\theta_3 = 1$), the percentage points are given in Tables 3–5.

The Bartlett and Cochran tests are thus extremely sensitive to deviations from normal in the measurement results (observed parameter). Correct use of those tests requires a knowledge of the distributions for the statistics for particular distributions for the observed random quantities. If the observed distribution differs from normal, one cannot apply the classical results. When the observed random quantities are closely described by distributions from the exponential family with a certain shape parameter θ_3, one can use the tables for the corresponding upper percentage points given here.

If regular checks are needed on the variance hypotheses for a certain particular model for the observed distribution, one can recommend the use of Monte Carlo simulation to derive under these conditions the distributions of the Bartlett or Cochran statistics followed by computer analysis of the regularities.

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