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COMPARATIVE ANALYSIS OF THE CRITERIA FOR CHECKING THE HYPOTHESIS OF UNIFORMITY OF LAW

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We examine a set of specific criteria designed to test hypotheses about the membership of observations to the uniform law. We studied the distribution of statistics of criteria and their power relative to various competing hypotheses. We identified advantages and disadvantages of the various criteria. It has been shown that a significant part of the criteria traditionally used when testing hypotheses about uniformity, is ectopic relative to a certain type of competing hypotheses. It is stressed that the special criteria for testing uniformity have no obvious advantages over non-parametric agreement criteria used for checking uniformity.
Keywords: *uniform law, hypothesis testing, statistical criterion, power of criterion.*

In applied mathematical statistics, the uniform distribution law of probabilities occupies an important position. Sometimes it is used as a model for the description of measurement errors of certain devices or or measurement systems.

The probability distribution function of the uniform law on the interval [0, 1] has the form $F(x) = x$. The hypothesis of membership of the sample $X_1, X_2, ..., X_n$ of independent observations of the random variable X to the uniform law can be written as H_0: $F(x) = x$, $x \in [0, 1]$. Most criteria for testing the hypothesis of uniformity in the specified range are based on estimates of order statistics of the magnitude X (on the elements $x_{(i)}$ of the variational series $0 < x_{(1)} < x_{(2)} < ... < x_{(n)} < 1$, constructed on $X_1, X_2, ..., X_n$). Below in expressions of the statistics of criteria, we use the notation $U_{(i)} = x_{(i)}$, $i = 1, ..., n$, $U_0 = 0$, $U_{n+1} = 1$.

As a rule, all criteria are focused on testing the *simple* hypothesis H_0 on the interval [0, 1]. In case it is necessary to test the membership hypothesis of the sample $X_1, X_2, ..., X_n$ to the uniform law in the interval $[a, b]$ (with shift parameter a and scale parameter $b - a$), to use all the uniformity criteria of the elements $x_{(i)}$ of the variational series $a < x_{(1)} < x_{(2)} < ... < x_{(n)} < b$, built on the sample $X_1, X_2, ..., X_n$, one transforms to the corresponding (required by criteria) order statistics as follows: $U_{(i)} = (x_{(i)} - a)/(b - a)$, $i = 1, ..., n$, $U_0 = 0$, $U_{n+1} = 1$. All the remaining order in applying the criteria for checking uniformity remains unchanged (as in the interval [0, 1]).

In verification of a *composite* hypothesis of uniformity of the form H_0: $F(x) = (x - a)/(b - a)$, $x \in [a, b]$, where a, b are unknown and must be found for the same sample, one proceeds as follows. For elements of a variational series $x_{(1)} < x_{(2)} < ... < x_{(n)}$, built on a sample $X_1, X_2, ..., X_n$, one finds estimates of the parameters

$$\hat{a} = x_{(1)} - (x_{(n)} - x_{(1)})/(n-1); \quad \hat{b} = x_{(n)} + (x_{(n)} - x_{(1)})/(n-1).$$

It is evident that verification of the composite hypothesis about the membership of $X_1, X_2, ..., X_n$ to the uniform law on the interval $[\hat{a}, \hat{b}]$, obtained for a given sample, is equivalent to the verification of the simple hypothesis of the membership of the partial sample of smaller size $n - 2$ (corresponding to the series $x_{(2)} < x_{(3)} < ... < x_{(n-1)}$) to the uniform law on the interval $[x_{(1)}, x_{(n)}]$, which corresponds to the sample range. In this case, in using all the considered criteria, the required values of order statistics are found in accordance with the relations

$$U_{i-1} = (x_{(i)} - x_{(1)})/(x_{(n)} - x_{(1)}); \quad i = 2, ..., n-1; \quad U_{n-1} = 1.$$

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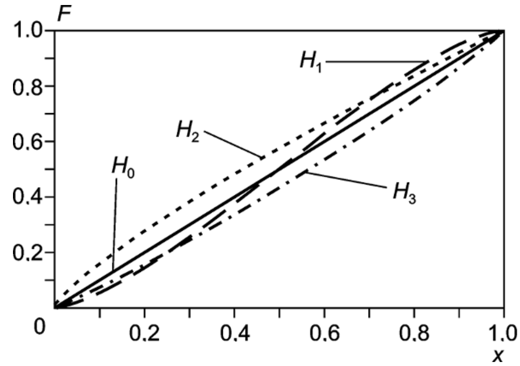

Fig. 1. Probability distribution functions corresponding to competing hypotheses.

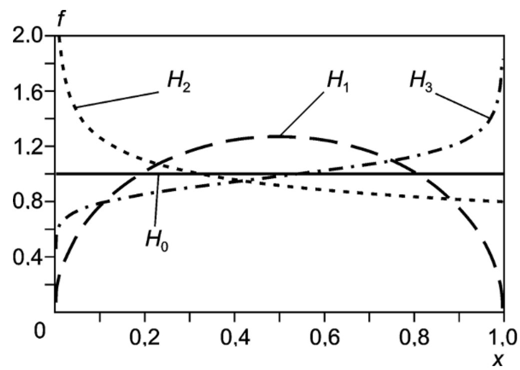

Fig. 2. Distribution density of laws corresponding to competing hypotheses.

To test the hypothesis that a sample belongs to a uniform law, we propose a set of statistical criteria, which can be divided into two subsets. The first consists of specific criteria, aimed at checking uniformity, and the second – a variety of fit tests, which can also be used to check the uniformity. The existence of multiple criteria poses practitioners a difficult choice since the available published information does not allow us to uniquely prefer some definite criterion.

This paper deals with the comparative analysis of specific criteria, in a subset of which one can identify three main groups of criteria of uniformity checks.

Statistical criteria of the first group include the use of differences of successive values of an ordered series

$$D_i = U_i - U_{i-1},$$

where $i = 1, ..., n + 1$; n is the sample size; $U_0 = 0$; $U_{n+1} = 1$.

The second group includes various modifications of the criteria for using the estimate differences between order statistics corresponding to the analyzed sample and, for example, of the expectations of these order statistics.

The third group includes the so-called entropy criteria based on various estimates of entropy.

Consideration of competing hypotheses. Testing hypotheses results in two types of errors: a type I error rejects the hypothesis H_0 when it is true; a type II error accepts (does not reject) the hypothesis H_0 when the competing hypothesis H_1 is valid. The level of significance α gives the probability of a type I error.

Typically, when using hypothesis testing criteria, one does not consider a specific competing hypothesis. In that case, when testing hypotheses about the form of the law it can be assumed that the competing hypothesis H_1 has the form: H_1: $F(x) \neq F(x, \theta_0)$. If the hypothesis H_1 is given and has the form H_1: $F(x) = F_1(x, \theta)$, then the specification of the quantity α to be used for the hypothesis testing criterion also determines the probability β of a type II error. An error of the second kind is that the hypothesis H_0 is not rejected when, in fact, the hypothesis H_1 is true. The power of the criterion is the difference $1 - \beta$. Obviously, the higher the power of the criterion used for a given value α, then the better it distinguishes the hypotheses H_0, H_1.

Of greatest interest is the capability of criteria to distinguish between similar competing hypotheses. In analyzing similar alternatives, one manages to find the finer points that characterize the properties of the criteria and to identify their fundamental advantages and disadvantages.

In this paper, we have studied the power of all the criteria examined with respect to three competing hypotheses that correspond to adherence of an observable random variable to the family of beta distributions of the 1st kind with density function

$$f(x) = \frac{1}{\theta_2 \mathrm{B}(\theta_0, \theta_1)} \left(\frac{x - \theta_3}{\theta_2} \right)^{\theta_0 - 1} \left(1 - \frac{x - \theta_3}{\theta_2} \right)^{\theta_1 - 1},$$

where $\mathrm{B}(\theta_0, \theta_1) = \Gamma(\theta_0)\Gamma(\theta_1)/\Gamma(\theta_0 + \theta_1)$ is the beta function; $\theta_0, \theta_1 \in (0, \infty)$ are shape parameters; $\theta_2 \in (0, \infty)$ is a scale parameter; $\theta_3 \in (-\infty, \infty)$ is a shift parameter; $x \in [\theta_3, \theta_3 + \theta_2]$.

We denote the beta-distribution of the 1st kind with specific parameter values as $\mathrm{B_I}(\theta_0; \theta_1; \theta_2; \theta_3)$. Then, the three competing hypotheses H_1, H_2, H_3 to be considered and sufficiently close to H_0, take the following form:

$$H_1\colon F(x) = \mathrm{B_I}(1.5; 1.5; 1; 0), x \in [0, 1];$$

$$H_2\colon F(x) = \mathrm{B_I}(0.8; 1; 1; 0), x \in [0, 1];$$

$$H_3\colon F(x) = \mathrm{B_I}(1.1; 0.9; 1; 0), x \in [0, 1].$$

The probability distribution functions corresponding to these hypotheses under consideration are shown in Fig. 1, and the density distributions in Fig. 2. It should be noted that the competing hypothesis H_1 corresponds to a law whose distribution function intersects the uniform distribution function; the distribution functions for H_2, H_3 lie above and below the uniform distribution function. The criteria have different capacities to distinguish among the hypotheses H_0, H_1 and among the hypotheses H_0, H_2, H_3.

Analysis of the power of criteria relative to H_1 revealed the inability of the individual criteria for small n samples and low levels of significance α to distinguish this hypothesis from H_0; that is, it showed the bias of the corresponding criteria (the power $1 - \beta$ turns out to be less than α). This insufficiency turns out to be characteristic not only of the specific criteria for testing uniformity, but also for most nonparametric agreement criteria [1].

Research results. As mentioned above, the set of specific criteria, whose statistics are presented in Table 1, can be divided into three groups according to their closeness to the criteria. The first group of criteria, using the differences between the elements of an ordered series include the criteria of Sherman [2, 3], Kimball [4], Moran 1 [5], Moran 2 [6], the refined Cressie criteria compared to [7] using expressions of the statistics $S_n^{(m)}$, $L_n^{(m)}$ [8], Pardo [9], and Swartz [10].

The second group of criteria, which considers the deviation of order statistics from their mathematical expectations (from the median, etc.), are the Hegazy–Green criteria with statistics T_1, T_2 [11], Frosini [12], Young [13], Cheng–Spiring [14], Greenwood [15], Greenwood–Quesenberry–Miller [16], Neyman–Barton with statistics N_2, N_3, and N_4 [17].

The third group includes the entropy criterion of Dudewics–van der Meulen [18] and two modifications whose statistics use other entropy estimates [19].

In carrying out this work, which is an extension of [20, 21], by using the methods of statistical modeling [22] we investigated the distribution statistics of all these criteria, expanded the table of percentage points, checked how the distribution of the normalized statistics is described by the corresponding asymptotic laws. We investigated the power of criteria with respect to different competing hypotheses, in particular with respect to H_1, H_2, H_3. It turns out that a number of the considered criteria are displaced with respect to the competing hypothesis H_1.

To prove the fact of displacement, we show in Figure 3 the distribution $G(\omega_n | H_i)$ of the statistics of the Sherman criterion corresponding to the validity of the hypotheses H_0, H_1 with sample sizes $n = 10$ and $n = 50$.

TABLE 1. Testing Criteria for Uniformity

Criteria	Statistic
Sherman	$\omega_n = \frac{1}{2}\sum_{i=1}^{n+1}\left\|U_i - U_{i-1} - \frac{1}{n+1}\right\|$
Kimball	$A = \sum_{i=1}^{n+1}\left(U_i - U_{i-1} - \frac{1}{n+1}\right)^2$
Moran 1	$B = \sum_{i=1}^{n+1}(U_i - U_{i-1})^2$
Moran 2	$M_n = -\sum_{i=1}^{n+1}\ln[(n+1)(U_i - U_{i-1})]$
Cheng–Spiring	$W_p = \left[(U_n - U_1)\frac{n+1}{n-1}\right]^2 \Big/ \sum_{i=1}^{n}(U_i - \bar{U})^2$
Young	$M = \frac{1}{l}\sum_{i=1}^{n}\min(D_i, D_{i+1}),\ D_1 = U_1,\ D_i = U_i - U_{i-1},\ D_{n+1} = 1 - U_n$
Greenwood	$G = (n+1)\sum_{i=1}^{n+1}(U_i - U_{i-1})^2$
Greenwood–Quesenberry–Miller	$Q = \sum_{i=1}^{n+1}(U_i - U_{i-1})^2 + \sum_{i=1}^{n}(U_{i+1} - U_i)(U_i - U_{i-1})$
Swartz	$A_n^* = \frac{n}{2}\sum_{i=1}^{n}\left(\frac{U_{i+1} - U_{i-1}}{2} - \frac{1}{n}\right)^2,\ U_0 = -U_1,\ U_{n+1} = 2 - U_n$
Hegazy–Green T_1	$T_1 = \frac{1}{n}\sum_{i=1}^{n}\left\|U_i - \frac{i}{n+1}\right\|$
Hegazy Green T_1^*	$T_1^* = \frac{1}{n}\sum_{i=1}^{n}\left\|U_i - \frac{i-1}{n-1}\right\|$
Hegazy–Green T_2	$T_2 = \frac{1}{n}\sum_{i=1}^{n}\left(U_i - \frac{i}{n+1}\right)^2$
Hegazy–Green T_2^*	$T_2^* = \frac{1}{n}\sum_{i=1}^{n}\left(U_i - \frac{i-1}{n-1}\right)^2$
Frosini	$B_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\|U_i - \frac{i-0.5}{n}\right\|$
Neyman–Barton, N_2	$N_2 = \sum_{j=1}^{2}V_j^2,\ V_j = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\pi_j(U_i - 0.5),\quad \pi_1(y) = 2\sqrt{3}y;\ \pi_2(y) = \sqrt{5}(6y^2 - 0.5)$

Criteria	Statistic
Neyman–Barton, N_3	$N_3 = \sum_{j=1}^{3} V_j^2,\ V_j = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \pi_j(U_i - 0.5), \quad \pi_3(y) = \sqrt{7}(20y^3 - 3y)$
Neyman–Barton, N_4	$N_4 = \sum_{j=1}^{4} V_j^2,\ V_j = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \pi_j(U_i - 0.5), \quad \pi_4(y) = 3(70y^4 - 15y^2 + 0.375)$
Dudewics–van der Meulen	$H(m,n) = -\frac{1}{n}\sum_{i=1}^{n} \ln\left\{\frac{n}{2m}(U_{i+m} - U_{i-m})\right\},\ m - \text{integer},\ m \le \frac{n}{2};$ if $i+m \ge n,\ U_{i+m} = U_n;$ if $i - m \le 1,\ U_{i-m} = U_1$
Modification 1 of entropy criterion	$HY_1 = -\frac{1}{n}\sum_{i=1}^{n} \ln\left(\frac{U_{i+m} - U_{i-m}}{\hat{F}(U_{i+m}) - \hat{F}(U_{i-m})}\right), \quad \hat{F}(U_i) = \frac{n-1}{n(n+1)}\left(i + \frac{1}{n-1} + \frac{U_i - U_{i-1}}{U_{i+1} - U_{i-1}}\right),\ i = 2,\ldots,n-1,$ $\hat{F}(U_1) = 1 - \hat{F}(U_n) = 1/(n+1)$
Modification 2 of entropy criterion	$HY_2 = -\sum_{i=1}^{n} \ln\left(\frac{U_{i+m} - U_{i-m}}{\hat{F}(U_{i+m}) - \hat{F}(U_{i-m})}\right)\left(\frac{\hat{F}(U_{i+m}) - \hat{F}(U_{i-m})}{\sum_{j=1}^{n}\left(\hat{F}(U_{j+m}) - \hat{F}(U_{j-m})\right)}\right)$
Pardo	$E_{m,n} = \frac{1}{n}\sum_{i=1}^{n} \frac{2m}{n(U_{i+m} - U_{i-m})}$
Cressie 1	$S_n^{(m)} = \sum_{i=0}^{n+1-m} \left(U_{i+m} - U_i - \frac{m}{n+1}\right)^2$
Cressie 2	$L_n^{(m)} = -\sum_{i=0}^{n+1-m} \ln\left[\frac{n+1}{m}(U_{i+m} - U_i)\right]$

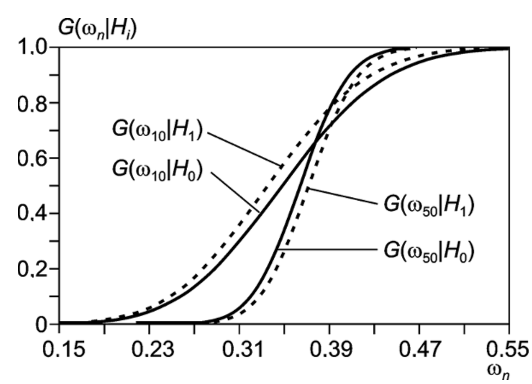

Fig. 3. The distribution $G(\omega_n|H_i)$ of the statistics of the Sherman criterion.

The right-sided criterion and verifiable hypothesis are rejected at elevated values of the statistic. The distribution of the statistic $G(\omega_{10}|H_1)$, which holds under the validity of H_1, is shifted relative to $G(\omega_{10}|H_0)$ not to the right, but to the left.

TABLE 2. Uniformity Criteria Ordered by the Power Relative to Competing Hypotheses H_1, H_2, H_3

Relative to H_1	$1-\beta$			Relative to H_2	$1-\beta$			Relative to H_3	$1-\beta$		
	$n = 10$	$n = 50$	$n = 100$		$n = 10$	$n = 50$	$n = 100$		$n = 10$	$n = 50$	$n = 100$
Modification 2 of entropy criterion	0.265	0.704	0.883	Hegazy–Green T_1	0.178	0.397	0.610	Hegazy–Green T_1	0.149	0.330	0.522
Neyman–Barton N_2	0.116	0.525	0.837	Frosini	0.171	0.389	0.603	Frosini	0.148	0.330	0.522
Cressie 2	0.201	0.644	0.820	Hegazy–Green T_2	0.175	0.389	0.602	Hegazy–Green T_1^*	0.145	0.327	0.520
Dudewics–van der Meulen	0.254	0.601	0.790	Neyman–Barton N_2	0.177	0.333	0.597	Hegazy–Green T_2	0.147	0.322	0.508
Modification 1 of entropy criterion	0.254	0.600	0.789	Hegazy–Green T_1^*	0.160	0.378	0.595	Hegazy–Green T_2^*	0.144	0.319	0.506
Neyman–Barton N_3	0.087	0.431	0.766	Hegazy–Green T_2^*	0.159	0.371	0.585	Neyman–Barton N_2	0.137	0.279	0.447
Neyman–Barton N_4	0.072	0.371	0.739	Neyman–Barton N_3	0.177	0.370	0.577	Neyman–Barton N_3	0.133	0.258	0.416
Cheng–Spiring	0.109	0.388	0.722	Neyman–Barton N_4	0.177	0.359	0.557	Neyman–Barton N_4	0.130	0.240	0.381
Swartz	0.154	0.398	0.583	Pardo	0.121	0.296	0.463	Pardo	0.118	0.208	0.291
Hegazy–Green T_1^*	0.101	0.226	0.443	Modification 1 of entropy criterion	0.097	0.189	0.328	Dudewics–van der Meulen	0.115	0.191	0.275
Hegazy–Green T_2^*	0.097	0.210	0.409	Dudewics–van der Meulen	0.097	0.188	0.327	Modification 1 of entropy criterion	0.115	0.191	0.275
Pardo	0.168	0.255	0.408	Cressie 1	0.157	0.239	0.314	Modification 2 of entropy criterion	0.114	0.185	0.267
Frosini	0.072	0.177	0.384	Modification 2 of entropy criterion	0.095	0.153	0.266	Cressie 2	0.123	0.170	0.226
Hegazy–Green T_1	0.054	0.133	0.322	Greenwood–Quesenberry–Miller	0.159	0.204	0.244	Cressie 1	0.118	0.167	0.218
Hegazy–Green T_2	0.060	0.137	0.308	Swartz	0.136	0.184	0.226	Swartz	0.129	0.172	0.206
Greenwood–Quesenberry–Miller	0.036	0.142	0.290	Cressie 2	0.120	0.140	0.217	Greenwood–Quesenberry–Miller	0.119	0.155	0.186
Kimball	0.059	0.160	0.279	Sherman	0.147	0.177	0.204	Kimball	0.116	0.142	0.165
Moran 1	0.059	0.160	0.279	Kimball	0.143	0.174	0.201	Moran 1	0.116	0.142	0.165
Greenwood	0.059	0.160	0.279	Moran 1	0.143	0.174	0.201	Greenwood	0.116	0.142	0.165
Sherman	0.065	0.147	0.215	Greenwood	0.143	0.174	0.201	Sherman	0.117	0.137	0.154
Cressie 1	0.037	0.082	0.187	Moran 2	0.150	0.172	0.193	Moran 2	0.116	0.131	0.143
Moran 2	0.072	0.135	0.187	Cheng–Spiring	0.106	0.134	0.168	Cheng–Spiring	0.100	0.105	0.106
Young	0.090	0.108	0.115	Young	0.108	0.108	0.108	Young	0.102	0.104	0.104

Consequently, the power $1 - \beta$ is less than the corresponding α. With increasing n, the bias disappears (cf. $G(\omega_{50}|H_0)$ and $G(\omega_{50}|H_1)$ in Fig. 3).

Expressions for the statistics of uniformity test criteria considered are shown in the Table 1. In Table 2, special uniformity verification criteria are presented in order of descending power relative to competing hypotheses H_1, H_2, H_3

(according the magnitude of the power $1 - \beta$, when $n = 100$ and a significance level $\alpha = 0.1$). Table 2 also shows the power estimation criteria for $n = 10$ and $n = 50$.

Some comparative analysis of the criteria of uniformity was carried out in [23]. In [19], the entropy criterion was compared with non-parametric agreement criteria. Various estimates of power can be found in other studies.

A typical disadvantage of most of the applied criteria for uniformity presented in Table 1 is the lack of information about the distribution statistics and the forced necessity for the use of tables of critical values (percentage points) that can be circumvented in principle with the existence of relevant software [8]. Another disadvantage of most of the criteria is the bias relative to competing hypotheses of type H_1 (for small sample sizes n and low levels of significance α). In Table 2, the dark color indicates criteria that for small n have a pronounced bias relative to the hypothesis H_1. To a lesser degree, a bias relative to H_1 is manifested in the criteria of Neyman–Barton with statistics N_2, N_3. This drawback is not marked only for certain criteria: for the entropy criterion of Dudewics–van der Meulen and its modifications, the criteria of Cheng–Spiring, Swartz, and Pardo.

All modifications of the criteria used in evaluating the various entropy statistics [18, 19] have shown a relatively high power relative to the competing hypothesis H_1. At the same time, relative to the hypotheses H_2, H_3 the estimate of the power of these criteria are more modest. It should be noted that only for these criteria, for small n, there are observed signs of bias relative to the hypothesis H_2.

The criterion of Neyman–Barton with N_2 statistics shows a high power relative to H_1 and comparatively good results relative to H_2, H_3. Consistently good ability to distinguish competing hypotheses from the uniform law is demonstrated by the criteria Hegazy–Green and Frosini. Low power is demonstrated by the criteria in whose statistics there is a summation of moduli or squared differences $U_i - U_{i-1}$ of successive values of order statistics (the criteria of Sherman, Kimball, Moran, Greenwood, Greenwood–Quesenberry–Miller). Especially low power relative to all three considered hypotheses is seen in the Yang criterion [13], which indicates a most unsuccessful attempt to use statistical criteria appropriate for checking the hypothesis of uniformity. It can be assumed that an equally unsuccessful idea would be to use such statistics in any criterion designed to test whether the observed sample agrees with some specific distribution law.

On the basis of studies of the properties of the set of criteria used for checking uniformity, we have prepared an application guide [8]. For checking a hypothesis about a test sample for adherence to some concrete distribution law there is developed a number of specific criteria; among this set, as a rule, there are criteria that are preferably applied to limited sample sizes and have a significant advantage in power, for example, compared with the general criteria of agreement. In this case (when checking for uniformity), such an advantage relative to non-parametric agreement criteria is not observed: stability is shown by the criteria of Zhang with statistics Z_A, Z_C and the criteria of Anderson–Darling test [8].

From the analysis of the properties of the entire set of criteria that can be used to test the hypothesis of adherence of a sample to a uniform law, it follows that the correct use of any one of the criteria for the formation of "reliable" statistical inference can often be inadequate. For greater objectivity of statistical inference, it is preferable to use a certain number of criteria that have definite advantages. Using a set of criteria, based on various measures of deviation of an empirical distribution from the theoretical, mproves the quality of statistical inferences.

A bias relative to certain proximate competing hypotheses for small n is exhibited not only for uniform criteria. There is a similar deficiency in some criteria used to verify normality [24–26].

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