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EXTENDING THE APPLICATION OF GRUBBS-TYPE TESTS IN REJECTING ANOMALOUS MEASUREMENTS

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Percentage-point tables have been compiled for statistics for tests of Grubbs type in testing for whether simultaneously three maximal or three minimal values represent excursions, and simultaneously the minimal and maximal values in the sample. Monte Carlo simulation has been applied to the distributions of the Grubbs test statistics as used in rejecting anomalous measurements for deviations of the observed law from normal.

Key words: *hypothesis testing, anomalous observations, Grubbs test, percentage points.*

Importance attaches to statistical tests for distinguishing anomalous measurements (outlying values). If one does not exclude such excursions from the data, then the classical methods of statistical analysis, which are usually not robust, often lead to incorrect conclusions.

Measurements that contain gross errors are usually quite obvious and can be discarded without using statistical methods. Statistical methods of detecting gross errors are desirable only in doubtful cases, when the information on the measurement quality is incomplete or unreliable [1], where one should check the obedience to assumptions under whose conditions it is correct to use them.

Most existing tests for rejecting doubtful data are based on the assumption that the observed random quantities follow a normal law. Examples are simple Grubbs tests [2–4] used for testing for anomaly (for estimating the anomaly) of outlying measurements. A standard [5] proposes the use of these tests, as this is the authentic test of the corresponding international standard ISO 5725. The statistics for the Grubbs test envisage checking for a sample containing either one outlying result (the least or largest) or two (two least or two largest).

Interest attaches to Grubbs tests because of the preference given to them in implementing standard ISO 5725. Here we show how distributions for the Grubbs test statistics alter in response to deviations of the observed law from normal. This illustrates what will occur with the use of the tests when the assumptions of normality are violated. Another purpose consists in using the tests with analogous statistics to check two hypotheses: for anomaly simultaneously in one minimal and one maximal element in the sample and for anomaly either in three minimal or three maximal elements in the sample.

The data have been obtained by Monte Carlo simulation based on suitable software.

Grubbs Test for One Outlying Value. Let $X_1, X_2, ..., X_n$ be the observed sample, while $X_{(1)} \le X_{(2)} \le ... \le X_{(n)}$ is the variational series constructed from it. The hypothesis H_0 being tested is that all the $X_1, X_2, ..., X_n$ belong to a common population. When one tests for excursion in the largest sample value, the competing hypothesis H_1 is that $X_{(1)}, X_{(2)}, ..., X_{(n-1)}$ follow one law and $X_{(n)}$ follows another one, substantially shifted to the right. In checking $X_{(n)}$ for an excursion, the Grubbs test takes the form

$$G_n = (X_{(n)} - \bar{X})/S, \tag{1}$$

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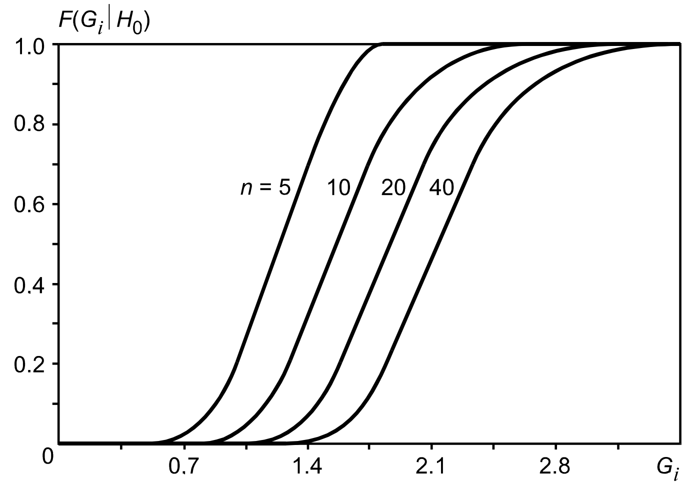

Fig. 1. Distributions of the (1) and (3) statistics for the Grubbs test with various sample volumes n (for a normal law).

where

$$\overline{X} = \frac{1}{n}\sum_{j=1}^{n} X_j ;$$

$$S^2 = \frac{1}{n-1}\sum_{j=1}^{n}(X_j - \overline{X})^2. \qquad (2)$$

In testing for an excursion in the least sample value, the competing hypothesis H_1 involves the assumption that $X_{(1)}$ belongs to some different law substantially displaced to the left. Then the statistic becomes

$$G_1 = (\overline{X} - X_{(1)})/S. \qquad (3)$$

The maximal or minimal element in the set is taken as an outlier if the value of the corresponding statistic exceeds the critical value: $G_n \geq G_{n,1-\alpha}$ or $G_1 \geq G_{1,1-\alpha}$, where α is the set significance level.

The (1) and (3) statistics are identically distributed. Figure 1 shows the forms of the conditional distributions $F(G_i/H_0)$ for the (1) and (3) statistics in relation to sample volume for normal distributions. The distribution for the statistic is substantially dependent on the sample volume n. The analytical forms of the statistic distributions are not given in the [5] standard or in [2–4]. They give only the upper percentage points for various sample volumes, since the decision on the anomaly in a minimal or maximal sample value is taken from the right-hand tail in the statistic distribution. In the standard, the percentage points are given only for sample volumes n from 3 to 40, while in [4] they are given for the range up to 147.

Table 5 for the percentage points in [5] gives the significance levels α incorrectly. In fact, this table gives values corresponding to significance levels of 0.005 (0.5%) and 0.025 (2.5%), not 0.01 (1%) and 0.05 (5%). This is shown clearly by [4]. This discrepancy was noted in research on the distributions of the Grubbs statistics. It still applies for the percentage points of statistics for analysis for outlying values simultaneously of two minimal or two maximal sample values. Consequently on this misunderstanding, when one specifies the significance level of 0.01 or 0.05 and uses the percentage points given in Table 5 of the standard [5], some of the outlying values will not be rejected.

That test can be used to distinguish anomalous results only for a normal distribution; if the observed random quantities belong to a different distribution, then the limiting distribution for the (1) and (3) statistics will be different. Figure 2

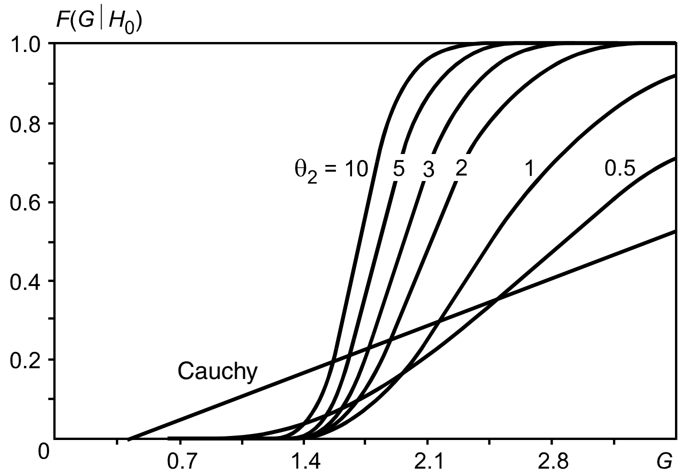

Fig. 2. Changes in the distributions of the (1) and (3) statistics for the Grubbs test in the case of differing laws from the family of (4) distributions with $n = 40$.

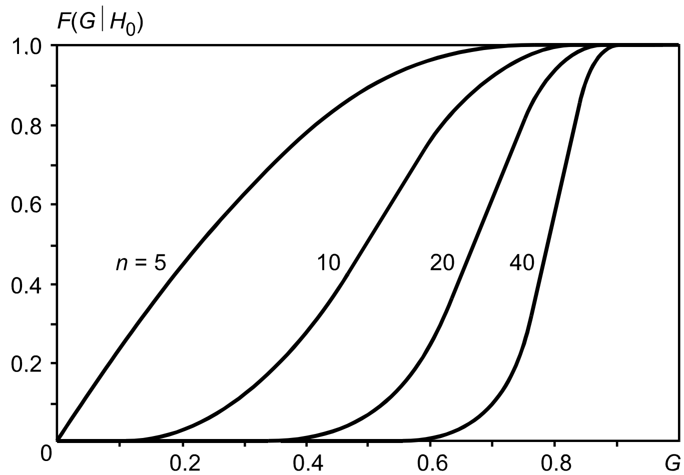

Fig. 3. Dependence of the distributions of statistics (5) and (9) for the Grubbs test on the sample volume (for a normal distribution).

shows the changes in the distributions of the (1) and (3) statistics when the random quantity belongs to various distributions in the exponential family with density

$$f(x) = \frac{\theta_2}{2\theta_1 \Gamma(1/\theta_2)} \exp\left\{ -\left(\frac{|x - \theta_0|}{\theta_1} \right)^{\theta_2} \right\}. \tag{4}$$

Particular cases of the family are a normal distribution with shape parameter $\theta_2 = 2$ and a Laplace distribution with $\theta_2 = 1$, while the limiting cases are a Cauchy distribution ($\theta_2 \to 0$) and a uniform one ($\theta_2 \to \infty$). Figure 2 gives the distributions for (1) and (3) with the observed laws of (4) for values of the shape parameter $\theta_2 = 0.5$, 1, 2, 3, 5, and 10 and also in the case of a Cauchy distribution with sample volume $n = 40$. It is clear that the distributions of the statistics differ very considerably.

The distribution of the statistic of the (1) form was first examined in [6]; in [1], a series of statistics of the forms of (1) and (3) is given, which differ one from another in combinations of known and estimated parameters for the shift and scale of the normal distribution. The use of tests based on these various statistics has been considered in [7, 8]. All these statistics, in spite of their similarities, differ from the Grubbs statistics of (1) and (3), in which estimators are used for both parameters of the normal distribution. Consequently, none of those statistics coincides precisely with the distribution of the Grubbs statistics of (1) and (3). In [9], there is a survey of some other tests for detecting and eliminating outlying sample values.

Checking for Two Outliers. In that case, the competing hypothesis H_1 may be related to the assumption that for example a different distribution applies for $X_{(n-1)}$ and $X_{(n)}$ (or $X_{(1)}$ and $X_{(2)}$). The Grubbs test in application for two largest values being simultaneously outliers takes the form

$$G = S^2_{n-1,n} / S^2_0, \tag{5}$$

where

$$S_0^2 = \sum_{j=1}^{n} (X_j - \overline{X})^2 ; \tag{6}$$

$$S^2_{n-1,n} = \sum_{j=1}^{n-2} (X_j - \overline{X}_{n-1,n})^2 ; \tag{7}$$

$$\overline{X}_{n-1,n} = \frac{1}{n-2} \sum_{j=1}^{n-2} X_j . \tag{8}$$

To check simultaneously the two least values $X_{(1)}$ and $X_{(2)}$ for being outliers, the statistic takes the form

$$G = S^2_{1,2} / S^2_0, \tag{9}$$

where

$$S^2_{1,2} = \sum_{j=3}^{n} (X_j - \overline{X}_{1,2})^2 ; \tag{10}$$

$$\overline{X}_{1,2} = \frac{1}{n-2} \sum_{j=3}^{n} X_j . \tag{11}$$

The two values ($X_{(n-1)}$, $X_{(n)}$ or $X_{(1)}$, $X_{(2)}$) are considered outliers if the value of the corresponding statistic is less than the critical value $G < G_\alpha$.

Figure 3 shows the forms of the conditional distributions $F(G \mid H_0)$ for the statistic G of (5) and (9) in relation to sample volume; the analytic forms for the distributions of G are not given in [5] or in [2–4]. They give only the lower percentage points for various sample volumes, since a decision on anomaly simultaneously for the two least or two largest sample values is taken on the left-hand tail of the statistic distribution. In the standard, the lower percentage points are given for sample volumes n only from 4 to 40. In [4] the lower percentage points for the distribution of statistic G are given for the range in n up to 149.

If the observed random quantities have a distribution different from normal, then the distributions for the (5) and (9) statistics take another form. With observed distributions of (4) form and values of shape parameter θ_2 = 0.5, 1, 2, 3, 4, 5,

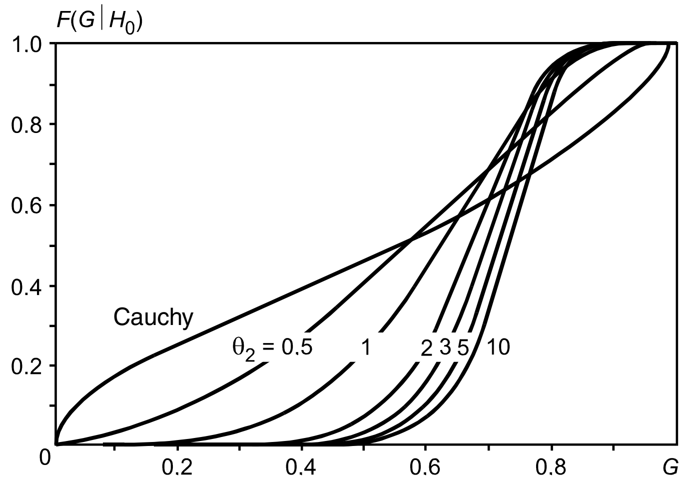

Fig. 4. Changes in the distributions of the (5) and (9) Grubbs statistics for various distributions with $n = 20$.

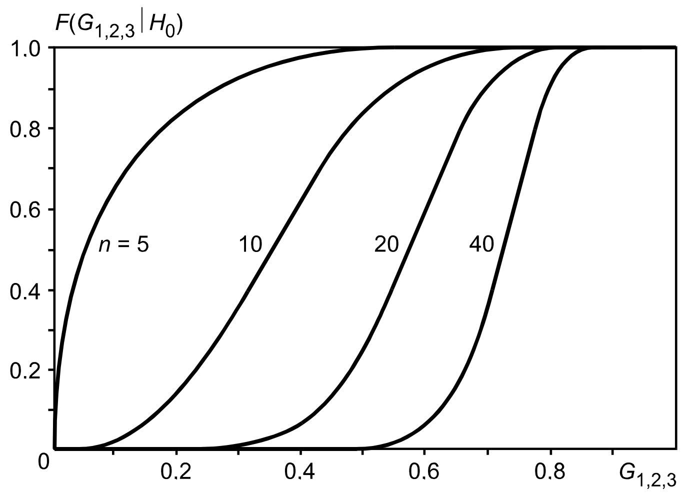

Fig. 5. Dependence of the distributions for the (12) and (13) statistics on sample volume (for a normal distribution).

and 10 and in the case of a Cauchy distribution, with sample volumes $n = 20$, these distributions take the form shown in Fig. 4. This implies that the distributions for the (5) and (9) Grubbs statistics are very much dependent on the distribution of the observed quantity.

As in the first case, a test with the (5)–(9) statistics can be used to discard anomalous observations by using the tables of percentage points given in [4, 5] only for a normal distribution. If the observed distribution differs from normal, Figs. 2 and 4 imply that the use of those tables can lead to one not only overlooking outliers, but also to the assignment as anomalous of data that are not such.

Checking for Three Outliers. The approach of (5)–(11) can be developed naturally to construct statistics for checking for anomaly simultaneously in three minimal or three maximal sample values, or else for checking for excursion simultaneously in the minimal and maximal values in the sample. One needs only to examine the distributions of the corresponding statistics.

When one checks for anomaly simultaneously in the three minimal or three maximal sample values, the competing hypothesis H_1 proposes that $X_{(1)}$, $X_{(2)}$, and $X_{(2)}$ (or $X_{(n-2)}$, $X_{(n-1)}$, and $X_{(n)}$) have some different distribution. The statistics for checking for anomaly simultaneously in the three minimal or three maximal sample values are formulated in accordance with

$$G_{1,2,3} = S_{1,2,3}^2 / S_0^2; \tag{12}$$

$$G_{n-2,n-1,n} = S_{n-2,n-1,n}^2 / S_0^2, \tag{13}$$

where

$$S_{1,2,3}^2 = \sum_{j=4}^{n} (X_j - \overline{X}_{1,2,3})^2 ; \quad \overline{X}_{1,2,3} = \frac{1}{n-3} \sum_{j=4}^{n} X_j ; \tag{14}$$

$$S_{n-2,n-1,n}^2 = \sum_{j=1}^{n-3} (X_j - \overline{X}_{n-2,n-1,n})^2 ;$$

$$\overline{X}_{n-2,n-1,n} = \frac{1}{n-3} \sum_{j=1}^{n-3} X_j . \tag{15}$$

The (12) and (13) statistics are identically distributed. All three measurements are considered as outliers if the value of the corresponding statistic is less than the critical one: $G_{1,2,3} < G_{3,\alpha}$ or $G_{n-2,n-1,n} < G_{3,\alpha}$.

Naturally, the distributions of these statistics are also dependent on n. Figure 5 shows the independence of the conditional distributions $F(G_{1,2,3} \mid H_0)$ and $F(G_{n-2,n-1,n} \mid H_0)$ for the (12) and (13) statistics when the sample is taken from a normal population. Table 1 gives the lower percentage points (α = 0.1; 0.5; 1; 5; 10%) for the distributions of the (12) and (13) statistics for $5 \le n \le 50$ as calculated by Monte Carlo simulation. Percentage points have been constructed from the simulated statistic samples. The volume of each sample from which the percentage points were estimated was 50000 simulated values. Table 1 gives values for the percentage points obtained by averaging over 15 such experiments.

The distributions for the (12) and (13) statistics are also dependent on the observed law. The distributions of them for laws of the form of (4) with shape parameters θ_2 = 0.5; 1; 2; 3; 5; 10 and in the case of a Cauchy distribution with sample volume $n = 20$ are given in Fig. 6.

Simultaneous Check for Excursions in the Least and Largest Values. The following formula gives the statistic for checking for anomaly simultaneously in the minimal and maximal sample values:

$$G_{1,n} = S_{1,n}^2 / S_0^2, \tag{16}$$

where

$$S_{1,n}^2 = \sum_{j=2}^{n-1} (X_j - \overline{X}_{1,n})^2 ; \quad \overline{X}_{1,n} = \frac{1}{n-2} \sum_{j=4}^{n-1} X_j . \tag{17}$$

Two values are considered as outliers for the given level of significance α if the value of the (16) statistic calculated on the sample is less than the critical value: $G_{1,n} < G_{1,n,\alpha}$.

Figure 7 shows the conditional distributions $F(G_{1,n} \mid H_0)$ for the (16) statistic $G_{1,n}$ in relation to sample volume n when the sample is taken from a normal population. Table 2 gives the lower percentage points for the distributions of the (12) statistic with $5 \le n \le 150$ as calculated by Monte Carlo simulation.

TABLE 1. Lower Percentage Points of the (12) and (13) Statistics for Tests of Grubbs Type

n	Lower percentage points for α in %					
	0.1	0.5	1	2.5	5	10
5	0.0000	0.0000	0.0000	0.0001	0.0004	0.0015
6	0.0002	0.0009	0.0019	0.0048	0.0099	0.0207
7	0.0023	0.0065	0.0106	0.0200	0.0332	0.0552
8	0.0079	0.0186	0.0268	0.0437	0.0640	0.0943
9	0.0176	0.0355	0.0478	0.0711	0.0966	0.1333
10	0.0314	0.0561	0.0717	0.1001	0.1302	0.1703
11	0.0471	0.0779	0.0968	0.1293	0.1619	0.2047
12	0.0659	0.1012	0.1222	0.1576	0.1925	0.2368
13	0.0841	0.1237	0.1471	0.1850	0.2206	0.2660
14	0.1035	0.1468	0.1707	0.2104	0.2475	0.2935
15	0.1234	0.1692	0.1943	0.2351	0.2726	0.3182
16	0.1412	0.1905	0.2170	0.2583	0.2962	0.3419
17	0.1607	0.2109	0.2374	0.2799	0.3178	0.3631
18	0.1797	0.2309	0.2583	0.3008	0.3382	0.3828
19	0.1973	0.2503	0.2782	0.3197	0.3575	0.4016
20	0.2161	0.2688	0.2966	0.3387	0.3757	0.4190
21	0.2313	0.2856	0.3139	0.3558	0.3924	0.4348
22	0.2488	0.3023	0.3303	0.3718	0.4082	0.4505
23	0.2643	0.3197	0.3466	0.3881	0.4238	0.4645
24	0.2795	0.3339	0.3606	0.4020	0.4375	0.4782
25	0.2952	0.3491	0.3762	0.4164	0.4510	0.4906
26	0.3091	0.3625	0.3890	0.4294	0.4638	0.5028
27	0.3209	0.3750	0.4022	0.4415	0.4756	0.5144
28	0.3357	0.3887	0.4151	0.4536	0.4874	0.5250
29	0.3475	0.4001	0.4270	0.4658	0.4984	0.5353
30	0.3608	0.4127	0.4382	0.4763	0.5087	0.5451
31	0.3710	0.4228	0.4486	0.4867	0.5186	0.5544
32	0.3797	0.4331	0.4596	0.4968	0.5282	0.5634
33	0.3935	0.4441	0.4692	0.5060	0.5370	0.5716
34	0.4040	0.4547	0.4793	0.5151	0.5456	0.5798
35	0.4131	0.4643	0.4885	0.5242	0.5541	0.5876
36	0.4239	0.4730	0.4974	0.5330	0.5623	0.5952
37	0.4317	0.4824	0.5064	0.5411	0.5697	0.6023
38	0.4414	0.4915	0.5149	0.5487	0.5772	0.6090
39	0.4511	0.4999	0.5228	0.5563	0.5843	0.6158
40	0.4610	0.5077	0.5296	0.5630	0.5910	0.6219
41	0.4667	0.5146	0.5381	0.5706	0.5978	0.6279
42	0.4751	0.5226	0.5452	0.5774	0.6041	0.6338
43	0.4839	0.5299	0.5517	0.5836	0.6102	0.6397
44	0.4910	0.5366	0.5585	0.5899	0.6159	0.6450
45	0.4997	0.5436	0.5651	0.5960	0.6217	0.6504
46	0.5057	0.5498	0.5713	0.6020	0.6274	0.6553
47	0.5131	0.5562	0.5775	0.6075	0.6327	0.6605
48	0.5191	0.5622	0.5833	0.6131	0.6380	0.6653
49	0.5247	0.5684	0.5891	0.6183	0.6430	0.6698
50	0.5316	0.5745	0.5947	0.6239	0.6477	0.6743

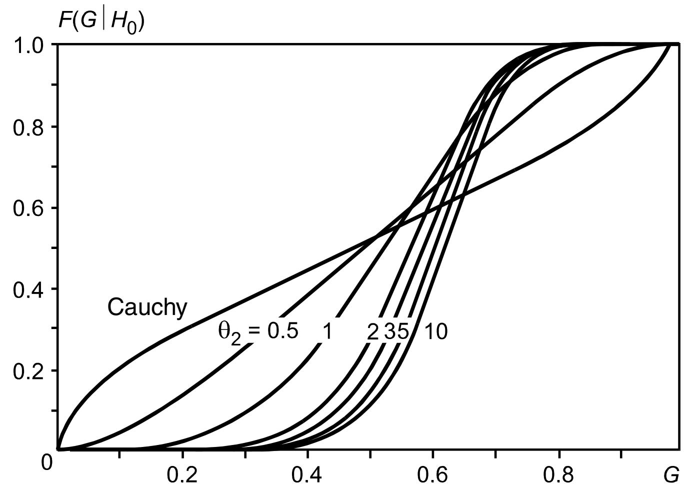

Fig. 6. Changes in the distributions of the (12) and (13) statistics for various laws in the (4) family with $n = 20$.

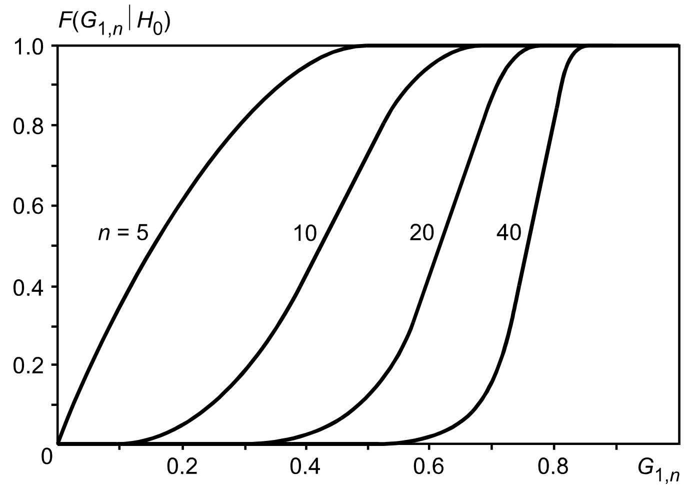

Fig. 7. Dependence of the distribution of the (16) statistic on sample volume (for a normal distribution).

The distribution of the (16) statistic is substantially dependent on the observed distribution. Figure 8 shows how the distribution for that statistic varies for observed distributions of (4) with values for the shape parameter $\theta_2 = 0.5; 1; 2; 3; 5; 10$ and in the case of a Cauchy distribution with sample volumes $n = 20$.

Each test enables one to reject outlying data present in the sample providing the number of outliers in it does not exceed the number for which the test is designed. When the test used corresponds to the actual number of outliers, the latter can usually be identified by the test. When the sample contains a number of gross errors larger than the statistic envisages, the test cannot distinguish them. For example, if a check for an outlier of the largest value alone does not give a positive result, that does not mean that that value is not an outlier. It is possible that in the sample there are more values that can be interpreted as anomalous. Their presence is reflected in the estimators for the variance of (2) and (6) and the estimators for the spread characteristics of (7), (10), (14), (15), and (17), since all of them are not robust. Consequently, when Grubbs tests are used one needs to successively test the sample for various numbers of gross errors.

TABLE 2. Lower Percentage Points of the (16) Statistic for a Test of Grubbs Type

n	Lower percentage points for α in %					
	0.1	0.5	1	2.5	5	10
5	0.0003	0.0012	0.0025	0.0063	0.0129	0.0265
6	0.0030	0.0089	0.0140	0.0262	0.0427	0.0698
7	0.0110	0.0243	0.0349	0.0562	0.0809	0.1178
8	0.0242	0.0468	0.0620	0.0908	0.1218	0.1644
9	0.0408	0.0712	0.0908	0.1252	0.1608	0.2073
10	0.0610	0.0991	0.1215	0.1606	0.1981	0.2464
11	0.0845	0.1279	0.1529	0.1939	0.2334	0.2821
12	0.1072	0.1544	0.1813	0.2247	0.2648	0.3135
13	0.1307	0.1813	0.2091	0.2538	0.2948	0.3428
14	0.1527	0.2065	0.2356	0.2808	0.3219	0.3696
15	0.1747	0.2313	0.2605	0.3059	0.3463	0.3936
16	0.1964	0.2537	0.2837	0.3291	0.3697	0.4160
17	0.2162	0.2756	0.3052	0.3512	0.3907	0.4367
18	0.2357	0.2969	0.3268	0.3718	0.4110	0.4556
19	0.2571	0.3164	0.3465	0.3912	0.4298	0.4730
20	0.2762	0.3358	0.3650	0.4094	0.4474	0.4895
21	0.2950	0.3543	0.3829	0.4264	0.4636	0.5051
22	0.3114	0.3702	0.3994	0.4424	0.4787	0.5191
23	0.3268	0.3864	0.4154	0.4573	0.4932	0.5326
24	0.3448	0.4013	0.4297	0.4714	0.5064	0.5451
25	0.3590	0.4153	0.4440	0.4848	0.5187	0.5567
26	0.3732	0.4294	0.4576	0.4973	0.5310	0.5679
27	0.3865	0.4423	0.4699	0.5097	0.5422	0.5784
28	0.3994	0.4547	0.4818	0.5208	0.5529	0.5884
29	0.4133	0.4673	0.4930	0.5317	0.5631	0.5978
30	0.4257	0.4791	0.5050	0.5422	0.5731	0.6067
31	0.4376	0.4885	0.5145	0.5511	0.5819	0.6152
32	0.4477	0.4995	0.5249	0.5608	0.5908	0.6235
33	0.4558	0.5099	0.5346	0.5702	0.5993	0.6314
34	0.4688	0.5189	0.5431	0.5783	0.6072	0.6384
35	0.4779	0.5285	0.5524	0.5864	0.6149	0.6456
36	0.4874	0.5374	0.5612	0.5946	0.6225	0.6525
37	0.4970	0.5459	0.5688	0.6022	0.6296	0.6591
38	0.5048	0.5540	0.5767	0.6091	0.6359	0.6652
39	0.5145	0.5617	0.5839	0.6166	0.6425	0.6711
40	0.5211	0.5692	0.5917	0.6229	0.6489	0.6768
41	0.5307	0.5767	0.5985	0.6295	0.6548	0.6823
42	0.5385	0.5835	0.6052	0.6360	0.6606	0.6877
43	0.5450	0.5902	0.6117	0.6417	0.6662	0.6928
44	0.5522	0.5970	0.6181	0.6476	0.6715	0.6977
45	0.5599	0.6033	0.6237	0.6529	0.6767	0.7025
46	0.5675	0.6090	0.6295	0.6582	0.6817	0.7071
47	0.5742	0.6154	0.6356	0.6637	0.6865	0.7115
48	0.5789	0.6211	0.6412	0.6687	0.6913	0.7159
49	0.5861	0.6270	0.6461	0.6733	0.6957	0.7200
50	0.5910	0.6324	0.6512	0.6783	0.7002	0.7240

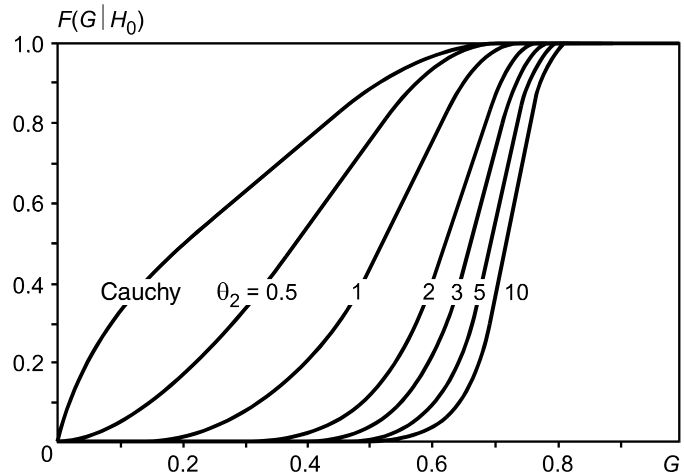

Fig. 8. Changes in the distribution of the (16) statistic for various laws from the family of (4) distributions for n = 20.

TABLE 3. Power of 1 – β Tests of Grubbs Type in Respect to a Mixture with 10% Symmetrical Corruption for n = 20

Significance level	Power of test with statistic		
	(1) and (3)	(5) and (9)	(16)
0.10	0.3763	0.3586	0.6094
0.05	0.3285	0.3115	0.5448
0.01	0.2431	0.2351	0.4164

Outliers in measurements may be due to displaced values related to the systematic error, and also to increase in the spread for various reasons. In the latter case, the outliers may relate to the least values or the largest ones. The capacity of these tests to identify anomalous results will be dependent on the form of the corruption.

As an example, we consider the power of the tests on a model with symmetrical corruption, in which a sample from a normal population with shift parameter μ and scale parameter σ is corrupted in 10% of the observations by a normal distribution with parameters μ and 5σ:

$$F(x) = 0.9F_{\text{Gauss}}(\mu, \sigma; x) + 0.1F_{\text{Gauss}}(\mu, 5\sigma; x).$$

The power of the test for a given probability α of errors of the first kind is determined by 1 – β, where β is the probability of errors of the second kind. In the present case, an error of the second kind is that an anomalous value is not identified as such. Table 3 gives the power of the criteria for checking for anomaly in a single minimal (or maximal) value in a sample, or simultaneously two minimal values (two maximal ones), or simultaneously one minimal one and one maximal one in a sample of volume n = 20. The higher power of the test with the (16) statistic in this case is due to the symmetry of the corruption.

Parametric Observation Rejection. The tables for the percentage points of Grubbs tests derived in [2–4] and the abbreviated table in [5] extend the tests considered here, and the tables constructed here for the corresponding percentage points enable one to reject gross errors correctly (outliers) in the case of a normal distribution. If the assumption of normality is violated, those percentage point tables cannot be used. We have seen above that the distributions of the statistics for tests of Grubbs type are substantially dependent on the true distribution of the observed random quantity. If necessary, there are no essential difficulties in constructing a model for the distribution of any particular statistic for a Grubbs-type test (or for determining the percentage points) with any distribution for the observed random quantities. The problem only is that there are too many distributions for which it would be desirable to have an effective procedure for rejecting anomalous measurements.

It is logical in analyzing data for anomalies to operate with the true distribution for the observed quantity. Then the rejection task is formulated as follows. The test hypothesis H_0 is that all the values $X_1, X_2, ..., X_n$ belong to the same population with distribution $F(x, \theta)$. When the largest sample value $X_{(n)}$ is tested for an outlier, the competing hypothesis H_1 is that the values $X_{(1)}, X_{(2)}, ..., X_{(n-1)}$ belong to $F(x, \theta)$, while $X_{(n)}$ belongs to a certain distribution $G(x)$, which is substantially displaced to the right relative to $F(x, \theta)$ e.g., $G(x) = F(x - A, \theta)$, where A is quite large. If $X_{(n)} \le d_{n,\alpha}$, then hypothesis H_0 is accepted, but otherwise H_1. If the null hypothesis

$$P\{\max_{1\le i\le n} X_i \le d_{n,\alpha}\} = [F(d_{n,\alpha})]^n = 1 - \alpha$$

is correct, the critical value is defined from $F(d_{n,\alpha}) = \sqrt[n]{1-\alpha}$.

When one checks for anomaly in the least value $X_{(1)}$, the hypothesis H_0 is adopted if $X_{(1)} \ge d_{1,\alpha}$; then

$$P\{\min_{1\le i\le n} X_i \ge d_{1,\alpha}\} = [1 - F(d_{1,\alpha})]^n = \alpha$$

and the critical value is defined by $F(d_{1,\alpha}) = 1 - \sqrt[n]{\alpha}$.

To identify correctly gross errors in a sample by means of this procedure, one must know the true distribution $F(x, \theta)$. However, the vector for the parameters θ in $F(x, \theta)$ often has to be estimated from the same sample, and consequently the rejection procedure is sometimes called parametric. Outliers in the sample are reflected in the estimators for the parameters $\hat{\theta}$. The resulting law $F(x, \hat{\theta})$ is substantially different from the true one, and consequently parametric methods of rejecting highly deviant observations become unstable [10].

A similar shortcoming applies to tests of Grubbs type: there is no guarantee that the sample does not contain more anomalous measurements than we examine for deviation. Then this may adversely affect the results.

In parametric rejection methods, one deals with this shortcoming by using robust estimation methods, e.g., maximum likelihood on grouped data [11], optimal *L* estimators on the sample quantiles [12, 13], and *MD* estimators. Robust estimation methods in parametric rejection make it very effective [11].

Robust methods can be used for estimating the mathematical expectation and standard deviation also in calculating statistics for Grubbs-type tests, but in that case it is essential to bear in mind that the estimation method is reflected in the distributions of the statistics.

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