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MODELS FOR STATISTICAL DISTRIBUTIONS IN NONPARAMETRIC FITTING TESTS ON COMPOSITE HYPOTHESES BASED ON MAXIMUM-LIKELIHOOD ESTIMATORS. PART II

B. Yu. Lemeshko and S. B. Lemeshko UDC 519.24

Revised results are given as tables of the percentage points and distribution models for nonparametric fitting tests involved in checking composite hypotheses in terms of families of gamma distribution and a two-sided exponential distribution when one uses maximum likelihood estimators.

Key words: *fitting tests, composite hypothesis testing, Kolmorogov test, Cramer–Mises–Smirnov test, Anderson–Darling test, gamma distribution, double exponential distribution.*

When one is testing composite hypotheses of the form H_0: $F(x) \in \{F(x, \theta), \theta \in \Theta\}$, where the estimator $\hat{\theta}$ for the scalar or vector parameter of the distribution $F(x, \theta)$ is calculated on the same set of data, the nonparametric fitting tests due to Kolmogorov, the ω^2 Cramer–Mises–Smirnov test, and the Ω^2 Anderson–Darling test lose the property of freedom from the distribution.

Here we give research results on the distributions of the statistics used in nonparametric fitting tests for use with composite hypotheses on the families of gamma distribution and two-sided exponential distribution (the paper is a conclusion of [1]). In the present case, the limiting distributions $G(S \mid H_0)$ for the fitting statistics are dependent not only on the form of $F(x, \theta)$ corresponding to hypothesis H_0 being accepted, and also on the type and number of the estimated parameters and the method of estimation, but also on the detailed value for the shape parameter.

When one tests hypotheses by means of the Kolmogorov test, it is recommended to use a statistic with the Bol'shev correction [2, 3] in the form [4]:

$$S_K = \left(6nD_n + 1\right) / 6\sqrt{n}, \tag{1}$$

where $D_n = \max(D_n^+, D_n^-)$ and

$$D_n^+ = \max_{1 \le i \le n} \left\{ i/n - F(x_i, \theta) \right\}; \quad D_n^- = \max_{1 \le i \le n} \left\{ F(x_i, \theta) - (i-1)/n \right\};$$

with n the sample volume, and $x_1, x_2, ..., x_n$ the elements of the sample arranged in increasing order.

In the ω^2 Cramer–Mises–Smirnov test, one uses a statistic of the following form [4]:

$$S_\omega = n\omega_n^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left\{ F(x_i, \theta) - (2i-1)/2n \right\}^2, \tag{2}$$

Novosibirsk State Technical University, Novosibirsk, Russia; e-mail: Lemeshko@fpm.ami.nstu.ru. Translated from Izmeritel'naya Tekhnika, No. 8, pp. 17–26, August, 2009. Original article submitted January 20, 2009.

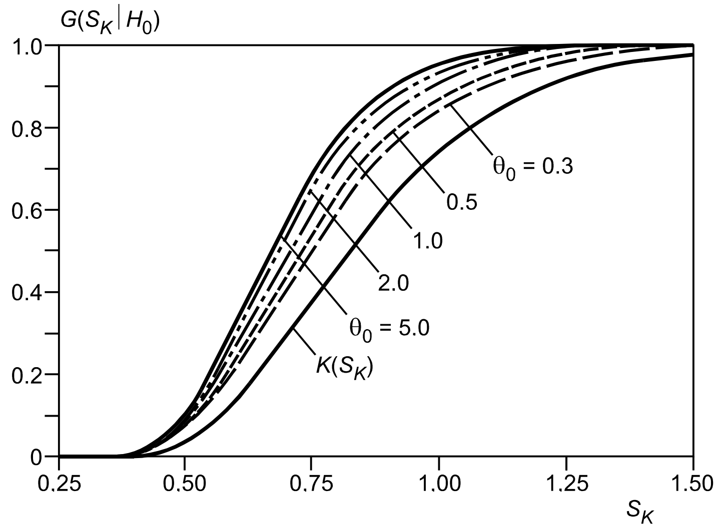

Fig. 1. Distributions of the statistics for the Kolmogorov test on considering composite hypotheses with respect to the gamma distribution and calculating maximum-likelihood estimators only for the scale parameter θ_1 in relation to the value of the shape parameter θ_0.

TABLE 1. Upper Percentage Points and Models for Limiting Distributions for the Kolmogorov Test

Shape parameter value	Estimated parameter	Percentage points			Model
		0.1	0.05	0.01	
0.3	Scale	1.096	1.211	1.444	B_3(6.6871; 4.8368; 4.4047; 1.9440; 0.281)
	Shape	0.976	1.070	1.262	B_3(6.4536; 5.7519; 3.3099; 1.6503; 0.280)
	Both parameters	0.905	0.990	1.162	B_3(6.9705; 5.6777; 3.6297; 1.5070; 0.270)
0.5	Ditto	1.051	1.160	1.379	B_3(6.9356; 5.0081; 4.3582; 1.8470; 0.280)
		0.961	1.052	1.236	B_3(6.3860; 5.9685; 3.1228; 1.6154; 0.280)
		0.884	0.965	1.131	B_3(6.4083; 5.9339; 3.2063; 1.4483; 0.2774)
1.0	»	0.994	1.095	1.299	B_3(6.7187; 5.3740; 3.7755; 1.6875; 0.282)
		0.936	1.022	1.191	B_3(6.1176; 6.4704; 2.6933; 1.5501; 0.280)
		0.862	0.940	1.097	B_3(5.6031; 6.1293; 2.7065; 1.3607; 0.2903)
2.0	»	0.952	1.044	1.228	B_3(5.8359; 22.6032; 2.1921; 4.00; 0.282)
		0.915	0.995	1.155	B_3(6.1387; 6.5644; 2.6021; 1.4840; 0.280)
		0.849	0.924	1.077	B_3(5.8324; 6.1446; 2.7546; 1.3280; 0.2862)
3.0	»	0.933	1.020	1.200	B_3(5.9055; 24.4312; 2.0996; 4.00; 0.282)
		0.906	0.985	1.140	B_3(6.1221; 6.6131; 2.5536; 1.4590; 0.280)
		0.845	0.919	1.070	B_3(6.0393; 6.1276; 2.8312; 1.3203; 0.2827)
4.0	»	0.923	1.008	1.181	B_3(5.9419; 27.1264; 1.9151; 4.00; 0.282)
		0.901	0.980	1.132	B_3(6.0827; 6.7095; 2.4956; 1.4494; 0.280)
		0.843	0.916	1.066	B_3(6.1584; 6.1187; 2.8748; 1.31704; 0.2807)
5.0	»	0.917	1.000	1.170	B_3(5.8774; 30.0692; 1.7199; 4.00; 0.282)
		0.899	0.977	1.127	B_3(6.0887; 6.7265; 2.4894; 1.4432; 0.280)
		0.842	0.915	1.063	B_3(6.1957; 6.1114; 2.8894; 1.3140; 0.2801)

TABLE 2. Upper Percentage Points and Models for Limiting Distributions for the Cramer–Mises–Smirnov Test

Shape parameter value	Estimated parameter	Percentage points			Model
		0.1	0.05	0.01	
0.3	Scale	0.233	0.305	0.482	B_3(3.2722; 1.9595; 16.1768; 0.750; 0.013)
	Shape	0.166	0.209	0.316	B_3(3.0247; 3.2256; 11.113; 0.7755; 0.0125)
	Both parameters	0.127	0.158	0.233	B_3(2.3607; 4.0840; 7.0606; 0.6189; 0.0145)
0.5	Ditto	0.205	0.264	0.413	B_3(3.2296; 2.1984; 14.3153; 0.700; 0.013)
		0.159	0.199	0.298	B_3(3.0143; 3.3504; 10.095; 0.7214; 0.0125)
		0.119	0.146	0.212	B_3(2.7216; 3.9844; 7.4993; 0.5372; 0.013)
1.0	»	0.175	0.220	0.336	B_3(3.1201; 2.5460; 11.1200; 0.600; 0.013)
		0.149	0.186	0.273	B_3(2.9928; 3.4716; 8.8275; 0.6346; 0.0125)
		0.111	0.136	0.194	B_3(3.0000; 3.8959; 7.3247; 0.4508; 0.012)
2.0	»	0.155	0.193	0.288	B_3(2.9463; 3.1124; 9.1160; 0.600; 0.013)
		0.142	0.176	0.256	B_3(2.9909; 3.5333; 8.2010; 0.5786; 0.0125)
		0.107	0.131	0.185	B_3(3.0533; 3.9402; 7.1173; 0.4246; 0.0118)
3.0	»	0.148	0.184	0.272	B_3(2.8840; 3.3796; 8.4342; 0.600; 0.013)
		0.139	0.172	0.251	B_3(2.9737; 3.5528; 7.8843; 0.5549; 0.0125)
		0.106	0.129	0.182	B_3(3.0703; 3.9618; 7.034; 0.4163; 0.0117)
4.0	»	0.145	0.179	0.264	B_3(2.8522; 3.5285; 8.1044; 0.600; 0.013)
		0.138	0.171	0.248	B_3(2.9677; 3.5426; 7.7632; 0.5418; 0.0125)
		0.105	0.128	0.180	B_3(3.0967; 3.9539; 7.064; 0.4122; 0.0116)
5.0	»	0.142	0.176	0.259	B_3(2.8249; 3.6280; 7.8756; 0.6000; 0.013)
		0.137	0.169	0.246	B_3(2.9638; 3.5465; 7.6558; 0.5334; 0.0125)
		0.105	0.128	0.179	B_3(4.4332; 3.6256; 10.552; 0.4098; 0.0084)

while in a test of Ω^2 Anderson–Darling type, one uses the form

$$S_\Omega = -n - 2\sum_{i=1}^{n}\left\{(2i-1)\ln F(x_i,\theta)/2n + \left(1-(2i-1)/2n\right)\ln(1-F(x_i,\theta))\right\}. \quad (3)$$

Model Refinement for Statistic Distribution for Nonparametric Tests of Fit in the Case of a Gamma Distribution. One may test composite hypotheses concerning a gamma distribution with density

$$f(x,\theta) = \left[x^{\theta_0-1}/\theta_1^{\theta_0}\Gamma(\theta_0)\right]\exp(-x/\theta_1),$$

where the limiting distributions $G(S|H_0)$ of the statistics for the nonparametric tests of fit are dependent on the particular value of the shape parameter θ_0. For example, Fig. 1 illustrates the dependence of the (1) Kolmogorov test statistic on the value of θ_0 in the testing of composite hypotheses when one calculates maximum-likelihood estimators (MLE) for the scale parameter of the gamma distribution (only one of the parameters).

Tables 1–3 give the upper percentage points derived by Monte Carlo simulation together with the models for the limiting statistic distributions for the Kolmogorov, Cramer–Mises–Smirnov, and Anderson–Darling tests respectively when MLE

TABLE 3. Upper Percentage Points and Models for Limiting Distributions for the Anderson–Darling Test

Shape parameter value	Estimated parameter	Percentage points			Model
		0.1	0.05	0.01	
0.3	Scale	1.300	1.655	2.543	B_3(3.3848; 2.8829; 14.684; 6.0416; 0.1088)
	Shape	1.021	1.258	1.865	B_3(3.1073; 3.7039; 8.6717; 4.3439; 0.1120)
	Both parameters	0.718	0.870	1.233	B_3(4.5322; 4.060; 10.0718; 2.9212; 0.078)
0.5	Ditto	1.183	1.490	2.260	B_3(5.0045; 2.9358; 18.8524; 5.2436; 0.077)
		0.993	1.221	1.791	B_3(3.1104; 3.7292; 8.0678; 4.0132; 0.1120)
		0.684	0.824	1.145	B_3(5.0079; 4.056; 10.0292; 2.5872; 0.073)
1.0	»	1.058	1.313	1.955	B_3(5.0314; 3.1848; 15.4626; 4.3804; 0.077)
		0.952	1.166	1.696	B_3(3.1149; 3.7919; 7.4813; 3.6770; 0.1120)
		0.657	0.785	1.084	B_3(5.0034; 4.1093; 9.1610; 2.3427; 0.073)
2.0	»	0.980	1.203	1.771	B_3(4.9479; 3.3747; 13.0426; 3.8304; 0.077)
		0.922	1.125	1.625	B_3(3.0434; 4.1620; 7.1516; 3.8500; 0.1120)
		0.643	0.766	1.051	B_3(4.9237; 4.2091; 8.6643; 2.2754; 0.073)
3.0	»	0.952	1.163	1.702	B_3(5.0367; 3.4129; 12.9013; 3.6867; 0.077)
		0.912	1.110	1.601	B_3(3.0565; 3.9092; 6.7844; 3.3972; 0.1120)
		0.639	0.761	1.043	B_3(4.9475; 4.2070; 8.6686; 2.2512; 0.073)
4.0	»	0.937	1.141	1.662	B_3(4.9432; 3.5038; 12.2240; 3.6302; 0.077)
		0.906	1.103	1.590	B_3(3.0531; 3.9437; 6.7619; 3.3993; 0.1120)
		0.637	0.758	1.039	B_3(4.9274; 4.2279; 8.5573; 2.2390; 0.073)
5.0	»	0.927	1.130	1.640	B_3(4.8810; 3.5762; 11.7894; 3.6051; 0.077)
		0.902	1.099	1.586	B_3(3.0502; 3.9640; 6.7510; 3.4024; 0.1120)
		0.636	0.757	1.037	B_3(4.9207; 4.2432; 8.4881; 2.2314; 0.073)

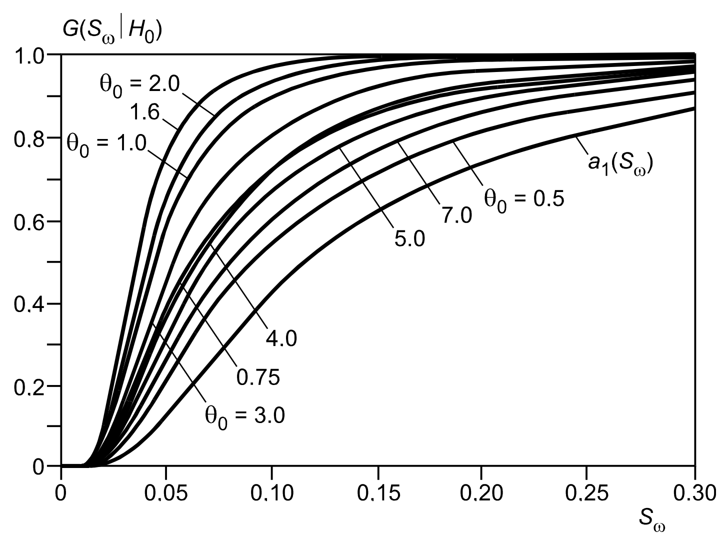

Fig. 2. Distributions of the statistic for the Cramer–Mises–Smirnov test in checking composite hypotheses relative to distributions of the (4) family on calculating maximum likelihood estimators for all three parameters in relation to values of the shape parameter θ_0.

TABLE 4. Upper Percentage Points and Limiting Distribution Models for Nonparametric Fitting Tests with $\theta_0 = 0.5$

Estimated parameter	Percentage points			Model
	0.9	0.95	0.99	
Kolmogorov test				
θ_0	1.184	1.322	1.596	γ(3.7437; 0.1349; 0.325)
θ_1	1.165	1.303	1.578	γ(3.5811; 0.1366; 0.325)
θ_2	1.182	1.308	1.560	γ(4.4361; 0.1186; 0.320)
θ_0, θ_1	1.123	1.259	1.534	γ(3.1115; 0.1442; 0.330)
θ_0, θ_2	1.144	1.271	1.528	γ(3.8417; 0.1265; 0.322)
θ_1, θ_2	1.110	1.233	1.480	γ(3.6713; 0.1251; 0.326)
θ_0, θ_1, θ_2	1.129	1.255	1.508	B_3(4.4961; 5.7241; 3.1229; 2.26825; 0.306)
Cramer–Mises–Smirnov test				
θ_0	0.325	0.441	0.723	B_3(2.6596; 1.5374; 22.6346; 1.100; 0.015)
θ_1	0.321	0.435	0.718	B_3(2.3196; 1.5425; 22.7256; 1.2000; 0.016)
θ_2	0.318	0.421	0.676	B_3(2.8412; 1.9552; 17.4052; 1.200; 0.014)
θ_0, θ_1	0.313	0.428	0.711	B_3(1.6693; 1.3771; 15.5765; 0.940; 0.017)
θ_0, θ_2	0.300	0.405	0.664	B_3(2.4600; 1.7966; 19.8161; 1.20; 0.014)
θ_1, θ_2	0.286	0.388	0.637	B_3(3.8085; 1.5324; 32.1564; 0.950; 0.011)
θ_0, θ_1, θ_2	0.295	0.399	0.656	B_3(3.0778; 1.6214; 30.1798; 1.2; 0.013)
Anderson–Darling test				
θ_0	1.735	2.303	3.697	B_3(5.1673; 1.7964; 33.1733; 6.000; 0.088)
θ_1	1.718	2.286	3.676	B_3(5.3595; 1.7388; 37.1241; 6.000; 0.087)
θ_2	1.819	2.335	3.617	B_3(3.4953; 2.2898; 14.9125; 6.400; 0.116)
θ_0, θ_1	1.671	2.238	3.633	B_3(5.786; 1.500; 45.3895; 5.200; 0.08)
θ_0, θ_2	1.631	2.159	3.454	B_3(3.1191; 2.0392; 20.4775; 6.600; 0.116)
θ_1, θ_2	1.578	2.093	3.356	B_3(3.0953; 2.0351; 22.1953; 6.800; 0.118)
θ_0, θ_1, θ_2	1.608	2.132	3.416	B_3(4.5039; 2.0396; 37.0448; 8.000; 0.092)

are used for the gamma distribution. In these cases, the distributions are well fitted by a family of beta distributions of type III with density function

$$B_3(\theta_0, \theta_1, \theta_2, \theta_3, \theta_4) = \frac{\theta_2^{\theta_0}}{\theta_3 \mathrm{B}(\theta_0, \theta_1)} \frac{\left((x-\theta_4)/\theta_3\right)^{\theta_0 - 1}\left(1-(x-\theta_4)/\theta_3\right)^{\theta_1 - 1}}{\left[1+(\theta_2 - 1)(x-\theta_4)/\theta_3\right]^{\theta_0+\theta_1}}.$$

These upper percentage points and distribution models (Tables 1–3) revise the results given in standardization recommendations [5]. For values of the shape parameter not coinciding with the tabulated ones, approximate values for the percentage points can be obtained by interpolation.

TABLE 5. Upper Percentage Points and Limiting Distribution Models for Nonparametric Fitting Tests with $\theta_0 = 0.75$

Estimated parameter	Percentage points			Model
	0.9	0.95	0.99	
Kolmogorov test				
θ_0	1.196	1.333	1.605	γ(3.7808; 0.1349; 0.330)
θ_1	1.172	1.309	1.584	B_3(4.5525; 4.9086; 3.8651; 2.3718; 0.315)
θ_2	1.068	1.173	1.384	B_3(4.7066; 10.8120; 1.8954; 2.50; 0.302)
θ_0, θ_1	1.126	1.263	1.560	B_3(4.0450; 4.9340; 3.7586; 2.3832; 0.310)
θ_0, θ_2	1.021	1.123	1.328	B_3(4.9912; 6.4499; 2.6816; 1.90; 0.295)
θ_1, θ_2	0.985	1.084	1.283	B_3(5.5451; 7.3578; 3.0559; 2.100; 0.280)
θ_0, θ_1, θ_2	0.937	1.032	1.223	B_3(4.5753; 6.8907; 2.74626; 1.8903; 0.294)
Cramer–Mises–Smirnov test				
θ_0	0.329	0.444	0.726	B_3(4.9844; 1.4891; 37.5211; 1.001; 0.0085)
θ_1	0.322	0.437	0.719	B_3(6.1042; 1.2892; 53.3676; 0.8800; 0.009)
θ_2	0.226	0.289	0.443	B_3(3.5628; 2.6431; 16.5587; 1.030; 0.010)
θ_0, θ_1	0.313	0.428	0.711	B_3(1.6779; 1.3775; 15.6587; 0.940; 0.017)
θ_0, θ_2	0.202	0.265	0.420	B_3(2.5230; 2.8292; 19.5602; 1.4650; 0.014)
θ_1, θ_2	0.192	0.255	0.408	B_3(2.6652; 2.4143; 24.7681; 1.300; 0.013)
θ_0, θ_1, θ_2	0.184	0.248	0.404	B_3(3.2636; 1.7846; 29.6713; 0.800; 0.0118)
Anderson–Darling test				
θ_0	1.755	2.322	3.715	B_3(5.5017; 1.7097; 32.6151; 5.4000; 0.09)
θ_1	1.721	2.290	3.681	B_3(5.7288; 1.7042; 38.1627; 5.700; 0.085)
θ_2	1.422	1.779	2.626	B_3(3.1406; 2.9653; 10.3579; 5.500; 0.12)
θ_0, θ_1	1.669	2.236	3.632	B_3(5.7330; 1.5217; 44.0784; 5.200; 0.078)
θ_0, θ_2	1.208	1.553	2.410	B_3(5.9765; 2.6769; 32.3123; 6.400; 0.070)
θ_1, θ_2	1.166	1.509	2.362	B_3(6.5437; 2.5007; 38.5262; 6.000; 0.07)
θ_0, θ_1, θ_2	1.116	1.465	2.322	B_3(6.2120; 2.1027; 40.3780; 4.800; 0.075)

Distribution Model Revision for Nonparametric Fitting Tests in the Case of a Two-Sided Exponential Distribution. When one tests composite hypotheses in terms of a law with the density

$$f(x,\theta) = \frac{\theta_0}{2\theta_1\Gamma(1/\theta_0)} \exp\left\{-\left(\frac{|x-\theta_2|}{\theta_1}\right)^{\theta_0}\right\}, \tag{4}$$

the distributions $G(S|H_0)$ of the statistics for the nonparametric fitting tests are dependent on the detailed value of the shape parameter θ_0.

Family (4) defines a wide range of symmetrical distributions, of which particular cases are a normal distribution for $\theta_0 = 2$ and a Laplace distribution for $\theta_0 = 1$. That distribution is sometimes called a two-sided exponential one [6], although

TABLE 6. Upper Percentage Points and Limiting Distribution Models for Nonparametric Fitting Tests with $\theta_0 = 1$

Estimated parameter	Percentage points			Model
	0.9	0.95	0.99	
Kolmogorov test				
θ_0	1.204	1.340	1.613	γ(3.9433; 0.1340; 0.3200)
θ_1	1.177	1.313	1.587	B_3(4.4680; 4.8450; 3.9105; 2.3784; 0.324)
θ_2	0.957	1.045	1.223	B_3(5.3541; 7.2519; 2.5630; 1.7652; 0.302)
θ_0, θ_1	1.130	1.268	1.545	B_3(3.9724; 4.8877; 3.7872; 2.3973; 0.3150)
θ_0, θ_2	0.911	0.995	1.162	B_3(4.9365; 8.1400; 2.2383; 1.7312; 0.3)
θ_1, θ_2	0.863	0.940	1.095	γ(6.2949; 0.0624; 0.2613)
θ_0, θ_1, θ_2	0.798	0.870	1.014	γ(5.5391; 0.0606; 0.2700)
Cramer–Mises–Smirnov test				
θ_0	0.333	0.447	0.7295	B_3(2.8981; 1.5614; 20.0694; 1.00; 0.014)
θ_1	0.323	0.438	0.719	B_3(3.9800; 1.4667; 38.0035; 1.13; 0.0111)
θ_2	0.152	0.187	0.268	B_3(3.3130; 3.8338; 10.0967; 0.7517; 0.011)
θ_0, θ_1	0.313	0.428	0.711	B_3(3.7712; 1.1413; 38.6694; 0.790; 0.011)
θ_0, θ_2	0.131	0.162	0.234	B_3(3.9062; 3.9000; 13.5396; 0.7491; 0.009)
θ_1, θ_2	0.115	0.144	0.213	B_3(4.4891; 3.7706; 17.5774; 0.7065; 0.0085)
θ_0, θ_1, θ_2	0.103	0.132	0.207	B_3(5.2856; 3.0510; 34.1638; 0.7312; 0.0079)
Anderson–Darling test				
θ_0	1.775	2.342	3.734	B_3(2.9208; 2.5613; 25.6028; 12.5850; 0.117)
θ_1	1.725	2.290	3.685	B_3(4.0842; 1.7532; 28.1434; 6.00; 0.105)
θ_2	1.071	1.302	1.837	B_3(4.2270; 3.0430; 8.4289; 3.000; 0.09)
θ_0, θ_1	1.668	2.235	3.630	B_3(3.7352; 1.5349; 29.4582; 5.300; 0.098)
θ_0, θ_2	0.871	1.062	1.522	B_3(4.8431; 4.1424; 14.2651; 4.6769; 0.073)
θ_1, θ_2	0.798	0.982	1.439	B_3(5.3576; 3.8690; 17.2148; 4.2386; 0.073)
θ_0, θ_1, θ_2	0.726	1.116	1.394	B_3(5.2973; 3.3781; 27.5085; 4.8145; 0.073)

as a rule in that case it implies $\theta_0 = 1$, or sometimes means an exponential family of distributions. However, in the last case, by this is meant a wider class of distributions.

There is a difference from the gamma distribution, for which the corresponding distributions shift to the region of lower values of the statistic as the shape parameter increases in that the behavior of the $G(S|H_0)$ distributions on testing hypotheses with respect to the family of (4) is such that with increase in θ_0 up to 1.64 these distributions shift to the right, while with further increase in θ_0 there begins a shift to the reverse direction (Fig. 2).

Tables 4–12 give the upper percentage points obtained by Monte Carlo simulation together with the models for the limiting distributions for the Kolmogorov, Cramer–Mises–Smirnov, and Anderson–Darling tests with shape parameters $\theta_0 = 0.5$; 0.75; 1; 1.6; 2; 3; 4; 5; 7 and the use of MLE parameters for the distributions of (4). These results revise and supplement those given in [7]. For values of the shape parameter not coinciding with the tabulated ones, approximate percentage points can be derived by interpolation.

TABLE 7. Upper Percentage Points and Limiting Distribution Models for Nonparametric Fitting Tests with $\theta_0 = 1.6$

Estimated parameter	Percentage points			Model
	0.9	0.95	0.99	
Kolmogorov test				
θ_0	1.216	1.351	1.621	B_3(4.2366; 5.7254; 2.8969; 2.4200; 0.330)
θ_1	1.185	1.322	1.596	B_3(4.3698; 5.2853; 3.3545; 2.3863; 0.318)
θ_2	0.851	0.923	1.069	B_3(5.4129; 7.6381; 2.1289; 1.3936; 0.290)
θ_0, θ_1	1.141	1.280	1.557	B_3(4.9730; 4.5743; 4.6422; 2.3576; 0.29)
θ_0, θ_2	0.828	0.898	1.039	B_3(6.2506; 7.4916; 2.5914; 1.4130; 0.275)
θ_1, θ_2	0.770	0.831	0.953	B_3(5.3623; 7.3149; 2.1379; 1.1702; 0.29))
θ_0, θ_1, θ_2	0.704	0.759	0.873	B_3(7.4853; 7.2752; 3.2095; 1.14609; 0.260)
Cramer–Mises–Smirnov test				
θ_0	0.339	0.453	0.735	*Sb*(3.6139; 1.0337; 3.400; 0.013)
θ_1	0.325	0.440	0.723	*Sb*(2.7348; 0.9148; 1.800; 0.016)
θ_2	0.121	0.149	0.219	B_3(4.5239; 3.7332; 15.6889; 0.6596; 0.009)
θ_0, θ_1	0.314	0.429	0.711	*Sb*(2.3111; 0.8115; 1.350; 0.016)
θ_0, θ_2	0.109	0.134	0.194	B_3(4.2190; 3.9949; 12.6139; 0.5642; 0.0087)
θ_1, θ_2	0.087	0.104	0.143	B_3(4.5491; 4.8658; 9.0448; 0.4000; 0.008)
θ_0, θ_1, θ_2	0.069	0.083	0.118	B_3(6.8750; 4.6392; 18.020; 0.3937; 0.006)
Anderson–Darling test				
θ_0	1.819	2.383	3.774	B_3(3.7982; 2.4042; 26.2612; 10.00; 0.095)
θ_1	1.735	2.304	3.697	B_3(3.6908; 2.1990; 32.1310; 10.00; 0.10)
θ_2	0.864	1.052	1.513	B_3(4.0782; 5.1594; 17.0570; 7.900; 0.09)
θ_0, θ_1	1.669	2.235	3.630	B_3(4.6625; 1.4267; 33.5120; 4.500; 0.09)
θ_0, θ_2	0.716	0.863	1.207	B_3(4.5576; 4.2326; 10.9573; 3.23142; 0.08)
θ_1, θ_2	0.589	0.695	0.941	B_3(4.5825; 5.3012; 7.9243; 2.5555; 0.0775)
θ_0, θ_1, θ_2	0.492	0.587	0.819	B_3(5.08840; 5.2459; 10.6760; 2.4738; 0.068)

The distributions of the test statistics are closely approximated by a family of type III beta distributions $B_3(\theta_0, \theta_1, \theta_2, \theta_3, \theta_4)$ and by a family of gamma distributions with density

$$\gamma(\theta_0, \theta_1, \theta_2) = \frac{1}{\theta_1^{\theta_0}\Gamma(\theta_0)}(x-\theta_2)^{\theta_0-1}e^{-(x-\theta_2)/\theta_1},$$

and by families of *Sb* Johnson distributions

$$Sb(\theta_0, \theta_1, \theta_2, \theta_3) = \frac{\theta_1\theta_2}{(x-\theta_3)(\theta_2+\theta_3-x)}\exp\left\{-\frac{1}{2}\left[\theta_0-\theta_1\ln\frac{x-\theta_3}{\theta_2+\theta_3-x}\right]^2\right\}$$

TABLE 8. Upper Percentage Points and Limiting Distribution Models for Nonparametric Fitting Tests with $\theta_0 = 2$

Estimated parameter	Percentage points			Model
	0.9	0.95	0.99	
Kolmogorov test				
θ_0	1.219	1.354	1.624	B_3(4.6934; 5.6544; 3.0971; 2.4099; 0.315)
θ_1	1.190	1.327	1.600	B_3(4.8849; 5.2341; 3.6279; 2.3872; 0.303)
θ_2	0.888	0.963	1.114	B_3(5.2604; 7.4327; 2.1872; 1.4774; 0.30)
θ_0, θ_1	1.148	1.287	1.564	B_3(4.6127; 4.8440; 4.1337; 2.4080; 0.295)
θ_0, θ_2	0.880	0.956	1.108	B_3(5.7052; 7.2179; 2.5877; 1.5433; 0.29)
θ_1, θ_2	0.835	0.909	1.057	γ(6.4721; 0.0580; 0.2620)
θ_0, θ_1, θ_2	0.784	0.861	1.021	B_3(9.3597; 5.7532; 5.8275; 1.4507; 0.2500)
Cramer–Mises–Smirnov test				
θ_0	0.341	0.456	0.737	*Sb*(2.7740; 0.9495; 1.9000; 0.0170)
θ_1	0.327	0.442	0.725	*Sb*(3.3182; 0.94801; 2.9500; 0.016)
θ_2	0.134	0.165	0.238	B_3(4.4331; 3.6365; 13.9198; 0.6632; 0.0084)
θ_0, θ_1	0.315	0.430	0.712	*Sb*(2.2458; 0.7970; 1.300; 0.017)
θ_0, θ_2	0.127	0.156	0.225	B_3(4.0430; 3.72568; 12.5794; 0.6313; 0.0087)
θ_1, θ_2	0.103	0.126	0.178	B_3(4.1153; 4.1748; 11.0347; 0.5116; 0.009)
θ_0, θ_1, θ_2	0.086	0.107	0.161	B_3(6.7594; 3.8575; 28.6668; 0.5921; 0.006)
Anderson–Darling test				
θ_0	1.842	2.404	3.796	B_3(3.0026; 2.7848; 21.7432; 12.5565; 0.111)
θ_1	1.743	2.309	3.704	B_3(3.4638; 2.3300; 35.7115; 12.6033; 0.105)
θ_2	0.892	1.087	1.552	B_3(4.1081; 5.0598; 16.9721; 7.9065; 0.09)
θ_0, θ_1	1.672	2.237	3.632	B_3(4.2125; 1.5874; 32.6127; 5.500; 0.09)
θ_0, θ_2	0.779	0.945	1.335	B_3(4.6827; 3.7977; 12.6413; 3.4486; 0.08)
θ_1, θ_2	0.630	0.750	1.032	B_3(4.7262; 4.6575; 9.4958; 2.7171; 0.0775)
θ_0, θ_1, θ_2	0.529	0.640	0.919	B_3(4.3857; 5.7110; 17.3440; 5.0052; 0.075)

or *Sl* Johnson distributions

$$Sl(\theta_0, \theta_1, \theta_2, \theta_3) = \frac{\theta_1}{(x-\theta_3)\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left[\theta_0 + \theta_1 \ln \frac{x-\theta_3}{\theta_2}\right]^2\right\}.$$

Example. We test a composite hypothesis that the sample of volume 200 observations given in [1] for the (4) family with shape parameter $\theta_0 = 5$ is used, for which MLE are employed for the parameters of scale $\hat{\theta}_1 = 0.9549$ and shift $\hat{\theta}_2 = -0.0025$.

The value of the statistic of (1) for the Kolmogorov test is $S_K^* = 0.8226$; calculations have been performed in accordance with the type III beta distribution B_3(4.5122; 5.6639; 2.8588; 2.0000; 0.310) (Table 11) attaining a significance level $P\{S_K > S_K^*\} \approx 0.362$.

TABLE 9. Upper Percentage Points and Limiting Distribution Models for Nonparametric Fitting Tests with $\theta_0 = 3$

Estimated parameter	Percentage points			Model
	0.9	0.95	0.99	
Kolmogorov test				
θ_0	1.222	1.357	1.626	B_3(6.5249; 5.0755; 4.5306; 2.4069; 0.285)
θ_1	1.197	1.334	1.606	B_3(5.2350; 5.0903; 3.9316; 2.3905; 0.300)
θ_2	0.998	1.095	1.291	B_3(8.5402; 6.1019; 4.4047; 1.8871; 0.250)
θ_0, θ_1	1.161	1.301	1.577	B_3(5.5689; 4.6553; 4.9339; 2.4147; 0.280)
θ_0, θ_2	0.999	1.096	1.293	B_3(6.5008; 7.8186; 3.25827; 2.1735; 0.270)
θ_1, θ_2	0.970	1.069	1.269	B_3(6.8503; 6.2212; 3.9819; 1.9216; 0.265)
θ_0, θ_1, θ_2	0.936	1.039	1.247	γ(3.6025; 0.10128; 0.3125)
Cramer–Mises–Smirnov test				
θ_0	0.345	0.459	0.741	B_3(3.2178; 1.6133; 19.2436; 1.000; 0.0125)
θ_1	0.330	0.445	0.727	B_3(3.6534; 1.5249; 28.5258; 1.0550; 0.0117)
θ_2	0.179	0.224	0.329	B_3(3.6203; 2.6395; 11.3638; 0.600; 0.010)
θ_0, θ_1	0.317	0.432	0.715	B_3(6.6688; 1.2016; 63.6672; 0.830; 0.008)
θ_0, θ_2	0.177	0.222	0.329	B_3(3.5065; 2.5837; 11.5972; 0.600; 0.010)
θ_1, θ_2	0.154	0.196	0.299	B_3(3.7581; 2.3887; 13.3525; 0.500; 0.010)
θ_0, θ_1, θ_2	0.138	0.181	0.289	*Sl*(1.1736; 1.2083; 0.1163; 0.0103)
Anderson–Darling test				
θ_0	1.881	2.441	3.835	B_3(3.7511; 2.3357; 19.6979; 8.0000; 0.095)
θ_1	1.757	2.324	3.718	B_3(4.1218; 2.1349; 30.0763; 8.500; 0.094)
θ_2	1.049	1.282	1.823	B_3(4.6108; 3.3193; 12.0931; 4.0000; 0.079)
θ_0, θ_1	1.679	2.245	3.638	B_3(6.0616; 1.6126; 47.4733; 5.800; 0.074)
θ_0, θ_2	0.989	1.215	1.741	B_3(4.7371; 3.2610; 13.7406; 4.0000; 0.070)
θ_1, θ_2	0.819	1.009	1.472	B_3(5.2098; 3.5915; 16.7524; 4.0000; 0.070)
θ_0, θ_1, θ_2	0.716	0.908	1.391	B_3(5.9548; 2.9777; 28.5342; 3.800; 0.069)

The value of statistic (2) for the Cramer–Mises–Smirnov test is $S_\omega^* = 0.1138$; the type III beta distribution B_3(4.1888; 1.9896; 21.3460; 0.7450; 0.009) (see Table 11) gives the significance level attained as $P\{S_\omega > S_\omega^*\} \approx 0.329$.

The value of the Anderson–Darling statistic (3) is $S_\Omega^* = 0.5773$; calculations in accordance with the III beta distribution of type B_3(4.1345; 3.0883; 22.0926; 6.800; 0.080) (Table 11) give the attained significance level $P\{S_\Omega > S_\Omega^*\} \approx 0.382$.

Then the hypothesis is not declined on all the tests, and evidently the (4) family with shape parameter $\theta_0 = 5$ is a more suitable model for that set than is a normal distribution (see example in [1]).

Conclusions. The following conclusions are drawn by combining the [1] results with the present study. More accurate models have been formulated for the statistic distributions for nonparametric fitting tests due to Kolmogorov, ω^2 Cramer–Mises–Smirnov, and Ω^2 Anderson–Darling in testing composite hypotheses on a series of distributions. These results refine

TABLE 10. Upper Percentage Points and Limiting Distribution Models for Nonparametric Fitting Tests with $\theta_0 = 4$

Estimated parameter	Percentage points			Model
	0.9	0.95	0.99	
Kolmogorov test				
θ_0	1.223	1.358	1.626	B_3(3.6243; 5.3291; 2.4503; 2.1853; 0.36)
θ_1	1.202	1.338	1.610	B_3(4.4775; 5.7536; 2.9612; 2.4028; 0.31)
θ_2	1.060	1.169	1.388	B_3(3.8031; 7.8639; 1.9955; 2.1337; 0.34)
θ_0, θ_1	1.172	1.311	1.586	B_3(2.6607; 6.1554; 2.0175; 2.4197; 0.364)
θ_0, θ_2	1.061	1.170	1.389	B_3(4.1178; 7.0193; 2.3554; 2.1116; 0.330)
θ_1, θ_2	1.039	1.150	1.372	B_3(4.4530; 6.5204; 2.8504; 2.1247; 0.315)
θ_0, θ_1, θ_2	1.013	1.126	1.353	γ(3.5001; 0.1150; 0.3200)
Cramer–Mises–Smirnov test				
θ_0	0.346	0.460	0.742	B_3(2.6493; 2.3780; 23.7392; 2.3027; 0.0133)
θ_1	0.332	0.447	0.729	B_3(2.9074; 1.7706; 24.9344; 1.40; 0.0134)
θ_2	0.212	0.270	0.409	B_3(3.2370; 2.7787; 15.5238; 1.05; 0.011)
θ_0, θ_1	0.319	0.434	0.717	B_3(2.8323; 1.4558; 24.26690; 1.0; 0.012)
θ_0, θ_2	0.212	0.271	0.412	B_3(2.9892; 2.7082; 14.1961; 1.0; 0.0117)
θ_1, θ_2	0.193	0.250	0.390	B_3(3.7333; 2.7350; 28.9872; 1.4094; 0.0094)
θ_0, θ_1, θ_2	0.178	0.236	0.381	B_3(3.5304; 2.1937; 29.8592; 1.000; 0.01)
Anderson–Darling test				
θ_0	1.899	2.458	3.853	B_3(2.7055; 3.0084; 16.8946; 12.5483; 0.12)
θ_1	1.771	2.338	3.729	B_3(2.6333; 2.6314; 22.5692; 12.5941; 0.125)
θ_2	1.188	1.467	2.118	B_3(2.7800; 5.1280; 11.7638; 10.5031; 0.11)
θ_0, θ_1	1.687	2.255	3.648	B_3(2.0354; 2.3209; 23.5136; 12.7679; 0.132)
θ_0, θ_2	1.153	1.432	2.090	B_3(3.6594; 3.4364; 13.5600; 5.9140; 0.084)
θ_1, θ_2	0.985	1.239	1.862	B_3(4.0113; 3.4057; 19.6395; 6.2684; 0.084)
θ_0, θ_1, θ_2	0.886	1.143	1.785	B_3(4.1564; 2.7774; 30.5627; 6.0165; 0.0822)

and extend the recommendations on standardization [5] and revise the models obtained in [7] for the statistics of nonparametric tests on verifying composite hypotheses with respect to the (4) distribution family.

Calculations have been performed on the percentage points and the corresponding models, which provide for correct use of nonparametric tests in statistical analysis with the use of MLE. There is marked dependence of the distributions on the estimation method [8], so these results cannot be used with other estimators, with the exception of cases where the asymptotic behavior of those estimators coincides with that of MLE.

When one tests composite hypotheses with respect to families of beta distributions of types I, II, and III, the distributions of the statistics for nonparametric fitting tests are dependent on the values of two shape parameters. Models have been constructed [9] for the distributions together with tables of the percentage points for various combinations of the two shape parameters (over 1500 models), and the research results have partially been published in [10].

TABLE 11. Upper Percentage Points and Limiting Distribution Models for Nonparametric Fitting Tests with $\theta_0 = 5$

Estimated parameter	Percentage points			Model
	0.9	0.95	0.99	
Kolmogorov test				
θ_0	1.223	1.357	1.626	B_3(3.4549; 6.2388; 2.0813; 2.300; 0.3600)
θ_1	1.205	1.342	1.613	B_3(3.1581; 6.2159; 1.9964; 2.300; 0.360)
θ_2	1.097	1.212	1.443	B_3(4.8171; 5.5295; 3.0757; 2.000; 0.320)
θ_0, θ_1	1.179	1.318	1.593	B_3(3.7224; 4.6425; 3.1224; 2.2000; 0.330)
θ_0, θ_2	1.097	1.213	1.444	B_3(4.9052; 5.6639; 2.9616; 2.0000; 0.310)
θ_1, θ_2	1.080	1.196	1.429	B_3(4.5122; 5.6639; 2.8588; 2.0000; 0.310)
θ_0, θ_1, θ_2	1.057	1.176	1.414	B_3(3.5446; 6.6218; 2.5197; 2.2850; 0.325)
Cramer–Mises–Smirnov test				
θ_0	0.347	0.460	0.742	B_3(3.3548; 1.7217; 20.2585; 1.1000; 0.012)
θ_1	0.334	0.448	0.731	B_3(3.2927; 1.6388; 23.4040; 1.100; 0.012)
θ_2	0.236	0.303	0.469	B_3(4.0012; 2.0310; 17.0057; 0.730; 0.0095)
θ_0, θ_1	0.321	0.436	0.719	B_3(4.0952; 1.3628; 33.3948; 0.900; 0.0095)
θ_0, θ_2	0.236	0.304	0.470	B_3(3.8227; 2.0270; 16.0637; 0.7200; 0.0095)
θ_1, θ_2	0.219	0.287	0.453	B_3(4.1888; 1.9896; 21.3460; 0.7450; 0.009)
θ_0, θ_1, θ_2	0.206	0.275	0.444	B_3(4.5253; 1.7162; 31.4699; 0.715; 0.009)
Anderson–Darling test				
θ_0	1.908	2.467	3.861	B_3(3.2750; 2.7257; 19.7022; 11.000; 0.105)
θ_1	1.782	2.349	3.740	B_3(3.7185; 2.2262; 24.9194; 8.500; 0.10)
θ_2	1.292	1.608	2.358	B_3(3.8528; 2.9989; 12.7999; 5.2000; 0.09)
θ_0, θ_1	1.696	2.264	3.658	B_3(3.9441; 1.9099; 34.2183; 8.000; 0.085)
θ_0, θ_2	1.271	1.588	2.346	B_3(3.6684; 3.0110; 13.4931; 5.550; 0.085)
θ_1, θ_2	1.110	1.411	2.153	B_3(4.1345; 3.0883; 22.0926; 6.800; 0.080)
θ_0, θ_1, θ_2	1.014	1.322	2.084	B_3(4.3601; 2.6164; 38.0670; 7.4729; 0.0785)

The power of a nonparametric test in verifying composite hypotheses as a rule is substantially higher (when MLE are used) than in checking simple hypotheses. Results have been given [11] from a comparison of power for fitting criteria (nonparametric and χ^2 types) for certain pairs of similar competing hypotheses in testing simple and composite ones, and these have been presented in more detail in [12, 13].

These results may reduce the number of errors arising in statistical analysis with the use of nonparametric fitting tests [14].

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TABLE 12. Upper Percentage Points and Limiting Distribution Models for Nonparametric Fitting Tests with $\theta_0 = 7$

Estimated parameter	Percentage points			Model
	0.9	0.95	0.99	
Kolmogorov test				
θ_0	1.222	1.357	1.625	B_3(3.4527; 6.2874; 2.062; 2.300; 0.3600)
θ_1	1.210	1.345	1.616	B_3(3.1789; 6.3997; 1.9239; 2.300; 0.360)
θ_2	1.137	1.137	1.502	B_3(4.4660; 5.47624; 2.6851; 2.000; 0.320)
θ_0, θ_1	1.190	1.328	1.601	B_3(3.8325; 4.7340; 3.0569; 2.2000; 0.330)
θ_0, θ_2	1.137	1.259	1.503	B_3(4.9890; 5.1511; 3.1470; 2.0000; 0.310)
θ_1, θ_2	1.124	1.247	1.493	B_3(4.5766; 5.2588; 2.9181; 2.0000; 0.310)
θ_0, θ_1, θ_2	1.107	1.232	1.480	B_3(3.5462; 6.6218; 2.2864; 2.2850; 0.325)
Cramer–Mises–Smirnov test				
θ_0	0.347	0.460	0.742	B_3(3.4065; 1.72170; 20.5769; 1.1000; 0.012)
θ_1	0.336	0.451	0.733	B_3(3.3961; 1.6388; 23.7205; 1.100; 0.012)
θ_2	0.265	0.345	0.542	B_3(4.0337; 1.7885; 18.1049; 0.730; 0.0095)
θ_0, θ_1	0.325	0.440	0.722	B_3(4.5574; 1.36280; 36.1643; 0.900; 0.0095)
θ_0, θ_2	0.266	0.346	0.545	B_3(4.1853; 1.7329; 19.4044; 0.7200; 0.0095)
θ_1, θ_2	0.253	0.333	0.531	B_3(4.3597; 1.7257; 23.2817; 0.7450; 0.009)
θ_0, θ_1, θ_2	0.241	0.323	0.522	B_3(4.3835; 1.5744; 28.6719; 0.715; 0.009)
Anderson–Darling test				
θ_0	1.916	2.475	3.864	B_3(3.3337; 2.7380; 19.7773; 11.000; 0.105)
θ_1	1.800	2.366	3.755	B_3(3.7496; 2.2445; 24.3153; 8.500; 0.10)
θ_2	1.431	1.804	2.698	B_3(3.9524; 2.6173; 14.0679; 5.2000; 0.09)
θ_0, θ_1	1.714	2.282	3.674	B_3(4.4259; 1.8843; 34.1400; 7.30; 0.085)
θ_0, θ_2	1.419	1.793	2.697	B_3(3.6688; 2.7003; 13.7324; 5.550; 0.085)
θ_1, θ_2	1.279	1.644	2.539	B_3(4.1773; 2.7020; 22.9667; 6.800; 0.080)
θ_0, θ_1, θ_2	1.188	1.560	2.470	B_3(4.5480; 2.1191; 28.5121; 5.000; 0.0785)

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