A Review of Tests for Exponentiality

Andrey P. Rogozhnikov, Boris Yu. Lemeshko
Novosibirsk State Technical University, Novosibirsk, Russia

Abstract – **A wide selection of tests for exponentiality is considered. Distributions of test statistics under true null hypothesis are studied and power of tests is estimated by means of methods of statistical simulation. A comparative analysis of power of tests with respect to competing alternatives with different shapes of hazard rate function is conducted. The conclusions are made on preference of one test or another under presence of specific competing alternatives.**

Index terms – **test, exponential distribution, power of test.**

I. Introduction

A NUMBER OF AUTHORS propose different statistical tests for testing a hypothesis of exponentiality. The wide variety of tests is caused by frequent application of the exponential model in applications. This is not least defined by that such a simple model makes it possible to solve problems basing upon analytical methods only.

Having a number of tests states a complicated problem of choice for practitioners as information available in publications does not definitely allow giving preference to some specific test. This is especially important when a problem arises of testing a hypothesis under presence of specific competing hypotheses. Of course, a set of goodness-of-fit tests could be applied but it appears from experience [1, 2] that the most powerful tests lie among the ones purposefully designed to test a hypothesis that sample follows one specific distribution.

A rather wide selection of tests for exponentiality is considered in some papers, e.g. [3, 4], and their power with respect to important competing hypotheses was studied by means of methods of statistical simulation. The results obtained made it possible to single out promising tests to apply in cases of having competing hypotheses with specific shape of hazard rate function and against wide class of competing hypotheses.

In this paper, some of the tests are excluded from consideration as they show unsatisfactory properties in important cases. Following [3], we excluded the tests of Epstein, Hartley, Deshpande ($J_{0.9}$), and Wong and Wong. Among the tests considered in [4] – $H_{m,n}$ entropy estimator-based test that shows low power and the tests of Henze and Meintanis with statistics $T_{n,a}^{(1)}$ and $T_{n,a}^{(2)}$ that show unexpectedly low power in several cases either.

We conducted a comparative analysis of tests from the promising group. In addition, we considered Bolshev's test for exponentiality proposed for testing hypothesis of exponentiality of observations in several small samples.

II. Problem Definition

Let $Exp(\theta)$ be exponential distribution with the density $f(x) = \exp(-x/\theta)/\theta$, $x \ge 0$, $\theta \equiv \lambda^{-1} > 0$, and $X_1, ..., X_n$ be given independent observations of nonnegative random variate. The composite hypothesis under test is H_0: X follows $Exp(\theta)$ under some value of θ.

In test statistics, we will use scaled observations $Y_j = X_j / \hat{\theta}_n$ or their transformed values $Z_j = 1 - \exp(-Y_j)$, $1 \le j \le n$, where $\hat{\theta}_n = \bar{X}_n = n^{-1} \sum_{j=1}^{n} X_j$ is the maximum likelihood estimator of parameter θ.

Let us denote order statistics of X_j, Y_j, and Z_j as $X_{(j)}$, $Y_{(j)}$, and $Z_{(j)}$ respectively. Denote $D_j = (n - j + 1)\left(X_{(j)} - X_{(j-1)}\right)$, $1 \le j \le n$, $X_{(0)} \equiv 0$.

III. Theory

A. *Gnedenko's F-test*

Gnedenko's *F*-test [3, 5, 6] is designed to test exponentiality against competing hypothesis H_1: distribution has monotone hazard rate. The test statistic is:

$$Q_R = \sum_{j=1}^{R} D_j / R \bigg/ \sum_{j=R+1}^{n} D_j / (n - R) .$$

Under true null hypothesis, Q_R has an *F* distribution with 2*R* and 2(*n*-*R*) degrees of freedom. H_0 is rejected for both small and large values of Q_R, concluding a decreasing hazard rate in the first case and an increasing – in the second. Our simulation with calculation of estimators of power of Gnedenko's test have shown that one should set *R*=[0.3*n*] out of *R*=[0.1*n*], [0.2*n*], ..., [0.9*n*] to maximize power when testing against hypotheses with monotone hazard rate.

B. Harris' modification of Gnedenko's F-test

This test was proposed by Harris [7] and discussed in [3] and [5]. The test statistic

$$Q'_R = \frac{\left(\sum_{j=1}^{R} D_j + \sum_{j=n-R+1}^{n} D_j\right) \Big/ 2R}{\sum_{j=R+1}^{n-R} D_j \Big/ (n-2R)}$$

follows *F*-distribution with 4*R* and 2(*n*-2*R*) degrees of freedom under true null hypothesis. The hypothesis is rejected for both small and large values of Q'_R.

We obtained that this test has sufficiently high power with respect to competing hypotheses with convex hazard rate and low power with respect to distributions with monotone hazard rate. The simulation conducted have shown that the test reaches its highest power with *R*=[0.1*n*].

C. Hollander and Proschan's test

The test of Hollander and Proschan [8, 3] is applied to one-sided alternatives with property "new better than used" ("new worse than used"). This property "may be interpreted as stating that the chance $\bar{F}(x)$ that a new unit will survive to age *x* is greater (less) than the chance $\bar{F}(x+y)/\bar{F}(y)$ than an unfailed unit of age *y* will survive an additional time *x*. That is, a new unit has stochastically greater life than a used unit of any age" [8]. The test statistic is:

$$T = \sum_{i>j>k} \psi\left(X_{(i)}, X_{(j)} + X_{(k)}\right),\ \psi(a,b) = \begin{cases} 1, a > b, \\ 0, a \le b. \end{cases}$$

The test is two-sided, authors give tables of approximate lower and upper critical values and the following normal approximation:

$$T^* = \left(T - \mathrm{E}[T \mid H_0]\right)\left(\mathrm{D}[T \mid H_0]\right)^{-1/2},$$

where $\mathrm{E}(T \mid H_0) = n(n-1)(n-2)/8$ and $\mathrm{D}[T \mid H_0] = 1.5n(n-1)(n-2) \times$ $\times\left[2(n-3)(n-4)/2592 + 7(n-3)/432 + 1/48\right]$.

D. Gini's test

This two-sided test with statistic

$$G_n = \sum_{j,k=1}^{n} \left|Y_j - Y_k\right| \Big/ 2n(n-1)$$

is considered in [9, 3, 4].

The asymptotic distribution of

$$G_n^* = \left[12(n-1)\right]^{1/2} \left\{G_n - 1/2\right\}$$

is standard normal [9] which, as we found, well describes G_n^* under $n \ge 10$. Gini's test is equivalent to the score test [4] with statistic $S_n = 2n - 2n^{-1}\sum_{j=1}^{n} jY_{(j)}$, which is connected to G_n with expression $(n-1)^{-1} S_n = 1 - G_n$. We considered the test based on G_n expressing the latter via S_n as it has lower computational complexity.

E. Tests based on empirical distribution function

E.1. Kolmogorov's test

In Kolmogorov's goodness-of-fit test, the value

$$D_n = \max\left\{ \max_{i \le j \le n}\left[\frac{j}{n} - Z_{(j)}\right], \max_{i \le j \le n}\left[Z_{(j)} - \frac{j-1}{n}\right]\right\}.$$

is used as a measure of difference between empirical distribution and the exponential law.

To decrease the dependence of the Kolmogorov's statistic on sample volume one should use the statistic with Bolshev's correction [10]:

$$K_n = \left(6n \cdot D_n + 1\right) / 6\sqrt{n}.$$

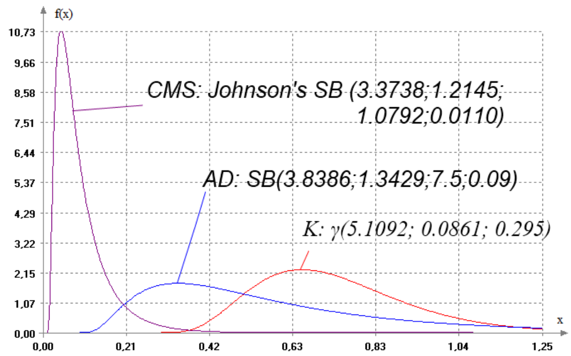

Fig. 1. Densities of distributions of statistics *K*, *CMS*, and *AD* under true null hypothesis.

E.2. Cramer–von Mises–Smirnov's test

The test statistic of Cramer–von Mises–Smirnov' test is:

$$CMS_n = \frac{1}{12n} + \sum_{j=1}^{n}\left(Z_{(j)} - \frac{2j-1}{2n}\right)^2.$$

E.3. Anderson–Darling's test

The statistic of Anderson–Darling's goodness-of-fit test for testing a sample for exponentiality is:

$$AD_n = -n - 2\sum_{j=1}^{n}\left[\frac{2j-1}{2n}\ln Z_j + \left(1 + \frac{2j-1}{2n}\right)\ln\left(1 - Z_j\right)\right].$$

The hypothesis of exponentiality is rejected by either Kolmogorov's, Cramer–von Mises–Smirnov's, or Anderson–Darling's test for large values of statistic.

A good model [11] for distribution of K_n under true complex null hypothesis and $n \ge 25$ is gamma distribution $\gamma(5.1092; 0.0861; 0.2950)$ with density

$$f(x) = \frac{1}{\theta_1^{\theta_0}\Gamma(\theta_0)}(x - \theta_2)^{\theta_0 - 1} e^{-(x-\theta_2)/\theta_1},\ x > \theta_2,$$

for CMS_n – Johnson's *SB* Distribution $Sb(3.3738; 0.2145; 1.0792; 0.011)$ with density

$$f(x) = \frac{\theta_1\theta_2}{\sqrt{2\pi}(x-\theta_3)(\theta_2+\theta_3-x)} \times$$
$$\times \exp\left\{-\frac{1}{2}\left(\theta_0 + \theta_1 \ln \frac{x-\theta_3}{\theta_2+\theta_3-x}\right)^2\right\}, \; x \in [\theta_3, \theta_3+\theta_2],$$

for AD_n – $Sb(3.8386; 1.3429; 7.500; 0.090)$ (see Fig.1).

F. Tests based on a characterization via the mean residual life function

X is distributed exponentially under the assumption $0 < \mu < \infty$ if, and only if $E(X - t \mid X > t) = \mu$ for each $t > 0$. This is equivalent to $E[\min(X,t)] = \mu F(t)$ for each $t > 0$ and, basing upon this, Baringhaus and Henze [12, 4] proposed Kolmogov and Cramer–von Mises–Smirnov type statistics.

The Kolmogorov type statistic of Baringhaus-Henze is:

$$\bar{K}_n = \sqrt{n} \sup_{t \geq 0} \left| \frac{1}{n}\sum_{j=1}^{n} \min(Y_j, t) - \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}\{Y_j \leq t\} \right| =$$
$$= \sqrt{n} \max(K_n^+, K_n^-),$$

where

$$K_n^+ = \max_{j=0,1,\ldots,n-1}\left[n^{-1}\left(Y_{(1)} + \ldots + Y_{(j)}\right) + Y_{(j+1)}(1 - j/n) - j/n\right],$$
$$K_n^- = \max_{j=0,1,\ldots,n-1}\left[j/n - n^{-1}\left(Y_{(1)} + \ldots + Y_{(j)}\right) - Y_{(j)}(1 - j/n)\right].$$

Here, it is as well reasonable to use the statistic with the Bolshev's correction:

$$K_n^* = \left(6n \cdot \bar{K}_n / \sqrt{n} + 1\right) \Big/ 6\sqrt{n}.$$

The Cramer–von Mises–Smirnov statistic of Baringhaus-Henze is:

$$CMS_n^* = n\int_0^\infty \left(\frac{1}{n}\sum_{j=1}^{n} \min(Y_j, t) - \frac{1}{n}\sum_{j=1}^{n} 1\{Y_j \leq t\}\right)^2 e^{-t} =$$
$$= n^{-1}\sum_{j,k=1}^{n}\left[2 - 3e^{-\min(Y_j, Y_k)} - 2\min(Y_j, Y_k) \times\right.$$
$$\left. \times \left(e^{-Y_j} + e^{-Y_k}\right) + 2e^{-\max(Y_j, Y_k)}\right].$$

The results of our simulation show that distributions of these two statistics do not match those given in [12], though one should not be surprised by this fact because the hypothesis under discussion is composite and involves calculation of MLE of scale parameter [13] (see also section IV).

G. Deshpande's test

The test was proposed in [14] and discussed in [3] for testing exponentiality against competing distributions with increasing hazard rate. The test statistic is calculated by

$$J_b = n(n-1)^{-1} \sum h_b(X_j, X_k),$$
$$\text{where } h_b = \begin{cases} 1, & X_j > bX_k, \\ 0, & otherwise, \end{cases}$$

and the sum is taken for all $1 \leq j,k \leq n$, $j \neq k$. When nothing is known about competing distribution a priori, one should use two-sided critical values.

Deshpande showed that $n^{1/2}(J_b - M(F))$ has asymptotically normal distribution with $\mu = 0$ and $\sigma^2 = 4\zeta_1$, where $M(F) = (b+1)^{-1}$ and

$$\zeta_1 = \frac{1}{4}\left\{1 + \frac{b}{b+2} + \frac{1}{2b+1} + \frac{2(1-b)}{b+1} - \frac{2b}{b^2+b+1} - \frac{4}{(b+1)^2}\right\}..$$

H. Cox and Oakes test

The hypothesis under test is rejected for both small and large values of test statistic

$$CO_n = n + \sum_{j=1}^{n} (1 - Y_j) \log Y_j.$$

The normalized statistic $CO_n\sqrt{6/n} \cdot \pi^{-1}$ has limit standard normal distribution.

The test with statistic CO_n is consistent against competing distributions with finite mathematical expectation and $E[X \log X - \log X] \neq 1$, provided the latter expectation exists.

I. Klar's test

The Klar's test [15, 4] is based upon the integrated distribution function and rejects the hypothesis of exponentiality for large values of statistic

$$KL_{n,a} = \frac{2(3a+2)n}{(2+a)(1+a)^2} -$$
$$-2a^3\sum_{j=1}^{n}\frac{\exp(-(1+a)Y_j)}{(1+a)^2} - \frac{2}{n}\sum_{j=1}^{n}\exp(-aY_j) +$$
$$+\frac{2}{n}\sum_{i<j}\left[a\left(Y_{(k)} - Y_{(j)}\right) - 2\right]\exp\left(-aY_{(j)}\right).$$

The author proposes [15] the use of a combined test that is based upon several statistics $KL_{n,a}$ with different values of a and rejects the hypothesis if at least one of $KL_{n,a}$ tests rejects it. Relying on simulation results, author concludes that the test $KL_n^{1,10}$ (combined of $KL_{n,1}$ and

$KL_{n,10}$) has the highest power with respect to alternatives of different types.

J. Bolshev's test

This test is designed for testing the hypothesis that a set of small samples follow exponential distributions [16].

Let $X_{i1},...,X_{in_i}$ $\left(n_i \ge 2;\ i=\overline{1,N}\right)$ be independent random variates. The hypothesis to test is H_0: X_{ij} follow exponential distributions with densities $a_i \exp(-a_i x)$ $\left(x>0,\ \ j=\overline{1,n_i};\ \ i=\overline{1,N}\right)$; the values of a_i are unknown and, possibly, different. If H_0 is true, the statistics $\zeta_{ir} = \sum_{j=1}^{r} X_{ij} \Big/ \sum_{j=1}^{r+1} X_{ij}$ $\left(r=\overline{1,n_i-1}\right)$ are independent and follow beta distributions with parameters r and 1 [16]. Consequently, statistics ζ_{ir}^{r} $\left(r=\overline{1,n_i-1};\ i=\overline{1,N}\right)$ are independent and identically uniformly distributed on [0,1]. One should apply non-parametric goodness-of-fit tests to test them for uniformity. In this paper, we used Anderson–Darling's test [17]. Total volume of small samples has a determinative effect on power of the Bolshev's test, thus we consider single samples without loss of generality.

K. Tests based upon empirical Laplace transform

In these tests, the Laplace transform $\psi(t)=E\left[\exp(-tX)\right]=\lambda\big/(t+\lambda)$ of exponential distribution is estimated by its empirical counterpart $\psi_n(t)=\dfrac{1}{n}\sum_{j=1}^{n}\exp\left(-tY_j\right)$.

K.1. The test of Baringhaus and Henze

In the test of Baringhaus and Henze [18, 4], the fact is used that ψ satisfies differential equation $(\lambda+t)\psi'(t)+\psi(t)=0$, $t\in R$. The test rejects the hypothesis for large values of statistic

$$BH_{n,a}=n^{-1}\sum_{j,k=1}^{n}\left[\frac{(1-Y_j)(1-Y_k)}{Y_j+Y_k+a}-\frac{Y_j+Y_k}{\left(Y_j+Y_k+a\right)^2}+.\right.$$
$$\left.+\frac{2Y_jY_k}{\left(Y_j+Y_k+a\right)^2}+\frac{2Y_jY_k}{\left(Y_j+Y_k+a\right)^3}\right].$$

The choice of a is proposed to be made according to a supposed competing hypothesis.

K.2. The test of Henze

The test of Henze [19, 4] rejects the hypothesis of exponentiality for large values of statistic

$$HE_{n,a}=n^{-1}\sum_{j,k=1}^{n}\left(Y_j+Y_k+a\right)^{-1}-$$
$$-\sum_{j=1}^{n}\exp\left(Y_j+a\right)E_1\left(Y_j+a\right)+n\left(1-a\exp(a)E_1(a)\right),$$

where $E_1(z)=\int_z^{\infty}t^{-1}\exp(-t)\,dt$ is exponential integral and $a>0$ is constant

K.3. The L-test of Henze and Meintanis

In the L-test of Henze and Meintanis [20, 4], the hypothesis is rejected for large values of statistic $L_{n,a}$. Description of $L_{n,a}$ distribution and tables of percent points for several a are given in [20].

$$L_{n,a}=\frac{1}{n}\sum_{j,k=1}^{n}\frac{1+\left(Y_j+Y_k+a+1\right)^2}{\left(Y_j+Y_k+a\right)^3}-2\sum_{j=1}^{n}\frac{1+Y_j+a}{\left(Y_j+a\right)^2}+\frac{n}{a}.$$

L. Tests based upon empirical characteristic function

L.1. W-tests of Henze and Meintanis

In the W-tests of Henze and Meintanis [21, 4], the null hypothesis is rejected for large values of statistics:

$$W_{n,a}^{(1)}=\frac{a}{2n}\sum_{j,k=1}^{n}\left[\frac{1}{a^2+\left(Y_j-Y_k\right)^2}-\frac{1}{a^2+\left(Y_j+Y_k\right)^2}\right.$$
$$\left.-\frac{4\left(Y_j+Y_k\right)}{\left(a^2+\left(Y_j+Y_k\right)^2\right)^2}+\frac{2a^2-6\left(Y_j-Y_k\right)^2}{\left(a^2+\left(Y_j-Y_k\right)^2\right)^3}+\frac{2a^2-6\left(Y_j+Y_k\right)^2}{\left(a^2+\left(Y_j+Y_k\right)^2\right)^3}\right],$$

$$W_{n,a}^{(2)}=\frac{\sqrt{\pi}}{4n\sqrt{a}}\sum_{j,k=1}^{n}\left[\left(1+\frac{2a-\left(Y_j-Y_k\right)^2}{4a^2}\right)\exp\left(-\frac{\left(Y_j-Y_k\right)^2}{4a}\right)+\right.$$
$$\left.+\left(\frac{2a-\left(Y_j+Y_k\right)^2}{4a^2}-\frac{Y_j+Y_k}{a}-1\right)\exp\left(-\frac{\left(Y_j+Y_k\right)^2}{4a}\right)\right].$$

L.2. Test for exponentiality of Epps and Pulley

As $n\to\infty$ the statistic of the test of Epps and Pulley [4]

$$EP_n=(48n)^{1/2}\left[\frac{1}{2}\sum_{j=1}^{n}\exp\left(-Y_j\right)-1/2\right]$$

is described by standard normal distribution; the null hypothesis is rejected for large values of $|EP_n|$. The test is consistent against competing distributions with monotone hazard rate, absolutely continuous CDF $F(x)$, $F(0)=0$, and $0<\mu<\infty$.

IV. Experimental Results

Some of the authors give normalizing transformations for test statistics, what makes it possible to apply standard normal law to normalized statistic to compute p-values while testing the hypothesis. In practice, such asymptotical results may turn to be unacceptable for samples of finite volume as a consequence of significant difference between distribution of specific statistic and its asymptotical model.

We used the methodology of statistical simulation [22] to verify how close actual distributions of statistics fit to corresponding theoretical models. The normalizing transformations were applied to statistics T, G, J_b, and CO when computing empirical distributions of test statistics under true null hypothesis. The results are based on 16'600 simulations, the true distribution was exponential with $\theta=1$: $F(x)=1-\exp(-x)$. The samples obtained were tested for fit with corresponding limit distributions by classical Kolmogorov's test. The *p*-values obtained in testing the simple hypothesis are given in Table I.

The results are following.

Application of limit distributions in the tests Q_R, Q'_R, *G*, *K*, *CMS*, *AD*, *B* is correct and makes it possible to accurately estimate the *p*-value.

Tests K^* and CMS^* are not delivered from the influence of sample volume on distribution of statistic. For $n \ge 20$, Johnson's SB distribution *Sb*(2.1275;1.6849;2.5437;0.26888) can serve as model for K^* and *Sb*(2.756;0.98223;1.8645;0.01602) – for CMS^*. When $n=10$ the use of these models leads to an underestimated *p*-value by K^* test and an overestimated *p*-value by CMS^*.

The normal approximation of distribution of statistic HP can be used only with limitations. Under $n \le 300$, computation of percent point tables would be the best choice. The use of asymptotical model is reasonable under $n \ge 400$.

In the test with statistic $J_{0.5}$, application of normal approximation do not lead to significant errors under $n \ge 50$; in tests with statistics EP and CO – under $n \ge 100$.

V. Discussion of Results

We compared the power of tests for relatively small sample volumes $n=20$ and $n=50$. Empirical distributions of test statistics under either true null hypothesis or competing hypotheses were found by 1'660'000 simulations. Null (exponential) distribution is characterized by constant hazard rate, thus we considered competing distributions that belong to three classes: with increasing, decreasing, and non-monotone hazard rates:

– Weibull $W(\theta)$ with $f(x)=\theta x^{\theta-1}\exp\left(-x^{\theta}\right)$;

– gamma $\Gamma(\theta)$ – $f(x)=\Gamma(\theta)^{-1}x^{\theta-1}\exp(-x)$;

– beta $\mathrm{B}(\theta_0,\theta_1)$ – $f(x)=\mathrm{B}(\theta_0,\theta_1)^{-1}x^{\theta_0-1}(1-x)^{\theta_1-1}$;

– uniform $U(0,1)$ on $[0,1]$;

– lognormal $LN(\theta)$ –

$$f(x)=\left(\theta x\sqrt{2\pi}\right)^{-1}\exp\left(-(\ln x)^2/2\theta^2\right);$$

– half-normal *HN* – $f(x)=(2/\pi)^{1/2}\exp\left(-x^2/2\right)$.

Distributions with increasing hazard rates are $W(\theta)$ and $\Gamma(\theta)$ $(\theta>1)$, $U(0,1)$, HN, $\mathrm{B}(1,2)$, $\mathrm{B}(2,1)$; decreasing – $W(\theta)$ and $\Gamma(\theta)$ $(\theta<1)$; non-monotone – LN, $B(0.5,1)$.

When computing critical values of statistics and estimators of power we assumed no prior knowledge of type of competing hypothesis. Therefore, we used two-sided critical regions in those tests that have a choice between left-sided and right-sided critical regions.

The estimators of power of tests with respect to different competing distributions with increasing, decreasing, and non-monotone hazard rates are given in Tables II, III, and IV respectively.

The tests with statistics BH and HE behave alike (this fact was mentioned in [4]), therefore below we will mention only the test with statistic BH. The choice of $a=0.5$ (and, correspondingly, $BH_{0.5}$) provides higher power compared to other values of a. In the *L*-test, statistic L_1 would be a reasonable choice in general case, in *W*-tests – statistic $W_1^{(1)}$, in *KL*-test – $KL^{1,10}$, obviously.

The following drawbacks should be mentioned in case of competing hypotheses with increasing failure rate (see Table II). Under $n=20$, the test of Bolshev is biased with respect to $W(1.2)$, $\Gamma(1.5)$, *HN*, and $\mathrm{B}(1,2)$ (i.e., its power is less than probability of type I error $\alpha=0.05$); the test $L_{0.1}$ is biased with respect to the same distributions and $W(1.4)$; the test $W_{2.5}^{(2)}$ is biased with respect to $W(1.2)$.

The test $Q'_{0.1}$ shows remarkably low power with respect to competing distributions with decreasing failure rate (see Table III).

In case of competing laws with non-monotone failure rate, the tests $W_{2.5}^{(2)}$ and BH_5 are biased with respect to $\mathrm{B}(0.5,1)$; the test $L_{0.1}$ – with respect to $LN(1)$ and $LN(0.8)$.

VI. Conclusion

Obviously, among the all tests studied, we cannot unambiguously choose a test with the highest power with respect to every considered competing hypothesis. It is as well unrealistic to place the tests in some unconditional order, e.g., descending by power.

In the same time, it is possible to select groups of tests equally promising in case of suggestion of certain kind of alternative.

Thus with respect to competing distributions with both increasing and decreasing failure rates, the tests of Cox and Oakes (*CO*), Anderson and Darling (*AD*), Henze and Meintanis (L_1 and $W_1^{(1)}$), Baringhaus-Henze ($BH_{0.5}$), and Henze ($HE_{0.5}$) show stably high power.

The tests of Harris ($Q'_{0.1}$) and Anderson and Darling (*AD*) possess high power with respect to alternatives with non-monotone hazard rates.

It is undesirable to use the tests of Harris $(Q'_{0.1})$, Bolshev (B), Henze and Meintanis ($L_{0.1}$ and $W_{2.5}^{(2)}$), Baringhaus and Henze $(BH_{0.5})$, and Henze $(HE_{0.5})$ under condition of small sample size or without specifying a concrete alternative (as a result of possible bias).

In a problem of choice of the most powerful test against given specified alternative beyond the ones considered in this paper, one should conduct a power research by similar methodology (and try different values of "fine-tuning" parameters in tests that have such). Of course, in such a research, knowledge of hazard rate function of the alternative should be taken into account.

The Bolshev's test (*B*) possess sufficiently high power with respect to laws with decreasing hazard rates but is inferior to the other tests in cases of other alternatives. One should keep in mind that the main advantage of the test is the approach that makes it possible to test the hypothesis of exponentiality of a set of small samples.

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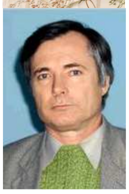
Andrey Pavlovich Rogozhnikov

Master of applied mathematics and computer science (2009), postgraduate student at the Chair of Applied Mathematics of NSTU



Boris Yurievich Lemeshko

Professor of Chair of Applied Mathematics of NSTU, Doctor of technical sciences, dean of the Faculty of Applied Mathematics and Computer Science of NSTU

TABLE I
P-VALUES IN TESTING GOODNESS-OF-FIT OF TEST STATISTIC DISTRIBUTIONS WITH CORRESPONDING THEORETICAL MODELS

n	$Q_{[0.3n]}$	$Q'_{[0.1n]}$	HP	G	K	K^*	CMS	CMS^*	AD	$J_{0.5}$	EP	CO	B
10	0.04	0.60	0.00	0.43	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.13
20	0.27	0.23	0.00	0.58	0.06	0.00	0.13	0.00	0.03	0.00	0.00	0.00	0.31
50	0.74	0.91	0.00	0.95	0.92	0.05	0.25	0.00	0.14	0.00	0.00	0.00	0.52
100	0.40	0.11	0.00	0.41	0.94	0.04	0.39	0.00	0.33	0.00	0.00	0.00	0.04
200	1.00	0.05	0.00	0.95	0.43	0.16	0.80	0.00	0.16	0.00	0.00	0.00	0.17
300	0.19	0.16	0.00	0.82	0.81	0.06	0.93	0.00	0.80	0.00	0.00	0.00	0.97
400	0.94	0.29	0.00	0.40	0.86	0.09	0.17	0.00	0.13	0.00	0.00	0.00	0.78
500	0.54	0.84	0.00	0.80	0.21	0.30	0.94	0.00	0.18	0.01	0.00	0.00	0.53

TABLE II
POWER OF TESTS FOR EXPONENTIALITY WITH RESPECT TO COMPETING HYPOTHESES WITH INCREASING FAILURE RATE $\times 1000$ (n=20, α=0.05).

	$W(1.2)$	$\Gamma(1.5)$	HN	B(1,2)	$W(1.4)$	$\Gamma(2)$	$W(1.5)$	$U(0.1)$	$\Gamma(4)$	B(2,1)
CO	138	217	191	220	381	551	527	528	996	999
$J_{0.5}$	123	186	177	200	322	466	448	545	984	998
EP	133	194	216	270	366	490	511	672	989	1000
G	130	187	216	277	356	473	498	714	987	1000
$Q_{0.3}$	107	148	163	180	265	358	367	445	926	985
$Q'_{0.1}$	54	78	60	73	93	176	126	120	692	607
HP	124	184	191	234	333	465	466	669	985	1000
K	119	169	178	204	290	407	398	528	961	994
K^*	152	204	244	303	358	462	480	729	975	1000
CMS	135	197	210	252	350	483	482	673	988	1000
CMS^*	134	191	221	279	358	477	496	716	988	1000
AD	109	168	170	209	307	451	438	628	987	1000
B	39	43	45	49	58	77	77	126	457	823
KL_1	127	179	219	288	347	450	485	731	982	1000
KL_{10}	101	175	110	107	270	464	377	244	986	958
$KL^{1,10}$	102	160	159	203	294	439	423	619	986	1000
$L_{0.1}$	10	12	20	21	27	61	50	48	661	673
$L_{0.75}$	135	217	177	192	367	552	507	473	996	998
L_1	140	220	192	214	380	554	523	526	996	999
$W_1^{(1)}$	132	194	189	221	328	465	450	661	982	1000
$W_{2.5}^{(1)}$	123	161	229	327	320	389	447	809	954	1000
$W_1^{(2)}$	118	150	228	342	303	356	424	829	923	1000
$W_{2.5}^{(2)}$	46	59	99	174	134	160	206	696	710	1000
$BH_{0.5}$	134	213	184	206	368	542	510	523	996	999
BH_1	140	213	205	240	381	537	526	597	995	1000
$BH_{1.5}$	138	206	211	253	378	520	523	631	993	1000
$BH_{2.5}$	131	192	210	263	363	489	507	662	989	1000
BH_5	115	166	196	257	327	435	466	679	979	1000
$HE_{0.5}$	139	219	192	214	379	552	522	529	996	999
HE_1	142	215	209	244	385	538	531	601	995	1000
$HE_{1.5}$	139	207	213	257	380	520	525	634	993	1000
$HE_{2.5}$	131	192	211	264	363	489	507	664	989	1000
HE_5	116	167	197	259	329	437	468	680	979	1000

TABLE III

POWER OF TESTS FOR EXPONENTIALITY WITH RESPECT TO COMPETING HYPOTHESES WITH DECREASING FAILURE RATE $\times 1000$ (n=20, α=0.05).

	Γ(0.7)	W(0.8)	Γ(0.5)	Γ(0.4)
CO	281	277	730	913
$J_{0.5}$	196	184	564	786
EP	200	236	543	759
G	203	239	547	759
$Q_{0.3}$	206	193	567	787
$Q'_{0.1}$	85	84	131	165
HP	226	216	601	811
K	156	173	470	706
K^*	112	134	380	617
CMS	178	199	525	756
CMS^*	185	218	523	748
AD	273	269	706	898
B	175	172	511	759
KL_1	184	223	505	723
KL_{10}	279	262	700	888
$KL^{1,10}$	272	279	686	879
$L_{0.1}$	363	309	785	933
$L_{0.75}$	254	259	670	869
L_1	240	252	645	851
$W_1^{(1)}$	155	162	464	693
$W_{2.5}^{(1)}$	155	191	429	638
$W_1^{(2)}$	148	184	404	606
$W_{2.5}^{(2)}$	182	235	428	611
$BH_{0.5}$	251	259	664	866
BH_1	225	248	614	827
$BH_{1.5}$	213	242	583	800
$BH_{2.5}$	202	238	548	765
BH_5	193	236	509	720
$HE_{0.5}$	243	255	650	855
HE_1	220	245	602	816
$HE_{1.5}$	210	240	575	791
$HE_{2.5}$	200	237	543	759
HE_5	192	235	508	719

TABLE IV

POWER OF TESTS FOR EXPONENTIALITY WITH RESPECT TO COMPETING HYPOTHESES WITH NON-MONOTONE FAILURE RATE $\times 1000$ (n=20, α=0.05).

	LN(1)	B(0.5,1)	LN(0.8)	LN(1.5)	LN(0.6)
CO	106	261	348	595	890
$J_{0.5}$	73	144	356	247	900
EP	132	66	259	663	801
G	117	60	246	659	801
$Q_{0.3}$	41	254	197	421	707
$Q'_{0.1}$	215	460	312	326	657
HP	61	137	307	314	863
K	138	154	304	572	851
K^*	122	117	287	549	841
CMS	152	188	341	616	891
CMS^*	151	112	279	652	841
AD	139	397	334	625	893
B	73	208	65	398	234
KL_1	150	71	231	662	759
KL_{10}	103	329	464	527	963
$KL^{1,10}$	149	290	347	652	914
$L_{0.1}$	5	530	42	389	430
$L_{0.75}$	104	214	399	602	939
L_1	109	171	375	619	923
$W_1^{(1)}$	105	226	350	513	888
$W_{2.5}^{(1)}$	133	100	173	631	640
$W_1^{(2)}$	128	94	147	618	549
$W_{2.5}^{(2)}$	193	41	91	680	291
$BH_{0.5}$	117	221	379	623	929
BH_1	125	137	334	646	892
$BH_{1.5}$	132	100	303	656	859
$BH_{2.5}$	142	69	266	666	809
BH_5	158	46	224	674	735
$HE_{0.5}$	110	175	372	623	921
HE_1	121	113	326	646	883
$HE_{1.5}$	129	87	298	656	851
$HE_{2.5}$	140	64	263	665	805
HE_5	156	45	224	673	735