Metrika (2005) 61: 29–45
DOI 10.1007/s001840400322

Metrika

Recent and classical tests for exponentiality: a partial review with comparisons

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Received: January 2002/Revised: January 2004

Abstract. A wide selection of classical and recent tests for exponentiality are discussed and compared. The classical procedures include the statistics of Kolmogorov-Smirnov and Cramér-von Mises, a statistic based on spacings, and a method involving the score function. Among the most recent approaches emphasized are methods based on the empirical Laplace transform and the empirical characteristic function, a method based on entropy as well as tests of the Kolmogorov-Smirnov and Cramér-von Mises type that utilize a characterization of exponentiality via the mean residual life function. We also propose a new goodness-of-fit test utilizing a novel characterization of the exponential distribution through its characteristic function. The finite-sample performance of the tests is investigated in an extensive simulation study.

AMS 1991 Classification Numbers: 62G10, 62G20

Key words: Goodness-of-fit test, exponential distribution, empirical characteristic function, empirical distribution function, integrated empirical distribution function, empirical Laplace transform, entropy, mean residual life function.

1. Introduction and summary

The assumption of exponentiality is heavily used in many modelling situations, particularly in life testing and reliability. Standard procedures for checking the validity of the exponential model are the tests of Kolmogorov-Smirnov and Cramér-von Mises, which utilize the empirical distribution function (EDF), procedures based on properties of normalized spacings and the score test. Recent years, however, have witnessed an increasing interest in using alternative methods, besides those directly involving the density and the distribution function of the exponential model, in constructing goodness-of-fit tests for exponentiality. These approaches include methods based on entropy and the Kullback-Leibler information and characterizations involving statistical transforms, such as the Laplace and the Fourier transform. Moreover, the old spirit of utilizing the EDF and the Kolmogorov-Smirnov and Cramér-von Mises distances has found new ways of expression.

In this article some of these tests are highlighted, discussed and compared. The tests under discussion are consistent against general alternatives, and/or have proved powerful among competing procedures in earlier studies. Consequently, some standard methods discussed in D'Agostino and Stephens (1986) such as the tests of Moran (1951) and Greenwood (1946) are not included. We also discarded many of the less powerful methods reviewed by Ascher (1990) and among recent procedures those constructed to guard against specific deviations from exponentiality (Alwasel (2001), Basu and Mitra (2002), Gatto and Jammalamadaka (2002), Klar (2000), Henze and Klar (2001), Muralidharan (2001) and Chaudhuri (1997)). A recent test of Morris and Szynal (2002) was not included because the test statistic is not invariant with respect to permutations of the data.

The paper is organized as follows. In Section 2 we introduce the test procedures, including a new test based on the empirical characteristic function, and discuss some of their features. Section 3 presents the results of an extensive Monte Carlo study on the power of the tests under consideration against a wide selection of popular alternatives to the exponential model.

2. The test statistics

Let $Exp(\theta)$ denote the exponential distribution with density $\exp(-x/\theta)/\theta$, $x \geq 0$. In what follows we consider goodness-of-fit tests for the class $\mathcal{E}xp = \{Exp(\theta) : \theta > 0\}$ of exponential distributions. Specifically, given a non-negative random variable X with density f, distribution function F and mean $\mu = \mathrm{E}(X)$, we wish to test the null hypothesis

$$H_0: \text{ The law of } X \text{ is } Exp(\theta) \text{ for some } \theta > 0,$$

against general alternatives, based on independent copies $X_1, \ldots, X_n$ of X. Since each statistic T_n considered will be a function of the scaled observations $Y_j = X_j/\hat{\theta}_n$ or their transformed values $Z_j = 1 - \exp(-Y_j)$, $1 \leq j \leq n$, where $\hat{\theta}_n = \bar{X}_n = n^{-1} \sum_{j=1}^n X_j$ is the maximum likelihood estimator of the parameter θ, it is scale invariant. As a consequence, the null distribution of T_n does not depend on the parameter θ.

In what follows, the order statistics of X_j, Y_j and Z_j will be denoted by $X_{(j)}$, $Y_{(j)}$ and $Z_{(j)}$, respectively. The notation $\xrightarrow{\mathcal{D}}$ means convergence in distribution, and $\mathbf{1}\{A\}$ denotes the indicator of an event A, which is 1 if A occurs and is 0, otherwise. The following tests are compared:

2.1. Tests based on the Empirical Distribution Function

These tests are based on direct measures of discrepancy between the EDF $F_n(x) = n^{-1} \sum_{j=1}^n \mathbf{1}\{Y_j \leq x\}$ of the scaled data $Y_1, \ldots, Y_n$ and the df of the unit exponential distribution. The most prominent EDF tests for exponentiality are the Kolomgorov-Smirnov and the Cramér-von Mises test. These reject the hypothesis H_0 for large values of

$$\begin{aligned} KS_n &= \sup_{x\geq 0} |F_n(x) - (1-\exp(-x))| \\ &= \max\left\{ \max_{1\leq j\leq n}\left[\frac{j}{n} - Z_{(j)}\right], \max_{1\leq j\leq n}\left[Z_{(j)} - \frac{j-1}{n}\right]\right\} \end{aligned}$$

and

$$\begin{aligned} \omega_n^2 &= \int_0^\infty (F_n(x) - (1-\exp(-x)))^2 \exp(-x)\, dx \\ &= \frac{1}{12n} + \sum_{j=1}^n \left(Z_{(j)} - \frac{2j-1}{2n}\right)^2, \end{aligned}$$

respectively (see D'Agostino and Stephens (1986), Sec. 4.9).

2.2. *Testing via the integrated Empirical Distribution Function*

Klar (2001) studied a test for exponentiality that is based on the weighted L^2-statistic

$$T_{n,a} = na^3 \int_0^\infty [\Psi_n(t) - \Psi(t)]^2 \exp(-at)dt,$$

where $\Psi(t) = \int_t^\infty (1 - F(x;1))dx = \exp(-t)$ and

$$\Psi_n(t) = \int_t^\infty (1 - F_n(x))dx = \frac{1}{n}\sum_{j=1}^n \max(Y_j - t, 0)$$

are the integrated survival function of the unit exponential distribution and its empirical counterpart, respectively, and $a > 0$ is a constant. A computationally convenient formula for $T_{n,a}$ is

$$\begin{aligned} T_{n,a} = &\frac{2(3a+2)n}{(2+a)(1+a)^2} - 2a^3 \sum_{j=1}^n \frac{\exp(-(1+a)Y_j)}{(1+a)^2} - \frac{2}{n}\sum_{j=1}^n \exp(-aY_j) \\ &+ \frac{2}{n}\sum_{j<k}[a(Y_{(k)} - Y_{(j)}) - 2]\exp(-aY_{(j)}). \end{aligned}$$

A test that rejects H_0 for large values of $T_{n,a}$ is consistent against each alternative distribution with finite positive expectation. The limit null distribution of $T_{n,a}$ is that of $a^3 \int_0^\infty \mathscr{W}^2(t)\exp(-at)dt$, where $\mathscr{W}$ is some centered Gaussian process in the Hilbert space of square-integrable functions on $(0,\infty)$.

Based on simulations Klar (2001) recommends to reject H_0 if at least one of the tests based on $T_{n,1}$ and $T_{n,10}$ rejects H_0. To give a formal description of this procedure, denoted by KL_n, let $\varphi_{a,\alpha} = \mathbf{1}\{T_{n,a} > z_{n,a}(\alpha)\}$ denote the level-α-test based on $T_{n,a}$. The test results in 1 (rejection) or 0 (no rejection). The combined level-α-test is $\varphi_{1,10}(\alpha) = \max(\varphi_{1,\beta}, \varphi_{10,\beta})$, where $\alpha/2 \le \beta \le \alpha$, and β is uniquely determined by the condition $\mathrm{E}[\varphi_{1,10}(\alpha)] = \alpha$ under H_0. In practice, β and the pertaining quantiles of $T_{n,1}$ and $T_{n,10}$ have to be found empirically by a search algorithm (see Klar (2001) for special numerical values) . Using the quantiles that belong to $\frac{\alpha}{2}$ leads to a conservative test.

2.3. A statistic based on spacings and the Gini index

In D'Agostino and Stephens (1986, p. 447), the following statistic based on the normalized spacings $D_j = (n+1-j)(X_{(j)} - X_{(j-1)})$ $(j = 1, \ldots, n; X_{(0)} = 0)$ is proposed: Calculate $U_k = \sum_{j=1}^{k} D_j / \sum_{j=1}^{n} X_j$, $1 \le k \le n-1$, which under H_0 is a sample of size $n-1$ from a uniform distribution in (0,1). Then a two-sided test is based on

$$S_n = \sum_{j=1}^{n-1} U_j = 2n - \frac{2}{n}\sum_{j=1}^{n} jY_{(j)}.$$

Since $(n-1)^{-1}S_n = 1 - G_n$, where

$$G_n = \frac{1}{2n(n-1)} \sum_{j,k=1}^{n} |Y_j - Y_k|$$

denotes the Gini index, a test based on S_n is equivalent to a two-sided test for exponentiality based on G_n. Such a test was proposed and thoroughly studied by Gail and Gastwirth (1978), who also stated the exact null distribution of G_n. The limit null distribution of $[12(n-1)]^{1/2}\{G_n - 1/2\}$ is standard normal. More generally, Hoeffding (1949) shows that, under the condition $\mathrm{E}(X^2) < \infty$, $\{G_n - \mathrm{E}(G_n)\}/\{Var(G_n)\}^{1/2}$ has a limiting standard normal distribution. Both Gail and Gastwirth (1978) and D'Agostino and Stephens (1986, p. 454) report that the test based on S_n resp. G_n has very high power. Notice that

$$\frac{S_n}{n-1} = 1 - G_n \to 1 - \frac{1}{2} \cdot \frac{\mathrm{E}|X_1 - X_2|}{\mathrm{E}(X)} \tag{2.1}$$

as $n \to \infty$ almost surely provided that $0 < \mathrm{E}(X) < \infty$. The limit in (2.1) is equal to $1/2$ if X has an exponential distribution. Since the same holds for other distributions (e.g., for the distribution having density $f(x) = 3/4$, if $0 \le x \le 1$, and $f(x) = 1/12$, if $1 \le x \le 4$; $f(x) = 0$, otherwise, a test for exponentiality based on S_n or G_n is not universally consistent.

2.4. Tests based on the entropy characterization

It is well known that among all distributions with density f concentrated on $[0, \infty)$ and fixed mean μ, the entropy $-\int_0^\infty f(x)\log f(x)dx$ is maximized by the exponential distribution. Grzegorzewski and Wieczorkowski (1999) and Ebrahimi et al. (1992) use this result to construct goodness-of-fit tests of exponentiality based on the entropy estimator

$$H_{m,n} = \frac{1}{n}\sum_{j=1}^{n} \log\left\{\frac{n}{2m}[X_{(j+m)} - X_{(j-m)}]\right\}$$

of Vasicek (1976). Here, m is an integer satisfying $1 \leq m < n/2$ and $X_{(j-m)} = X_{(1)}$, $X_{(j+m)} = X_{(n)}$, if $j - m \leq 0$ or $j + m \geq n$, respectively. Rejection of H_0 is for small values of $H_{m,n}$.

Ebrahimi et al. (1992) show that, if $m, n \to \infty$, and $m/n \to 0$, their test is consistent against distributions with finite mean, while Grzegorzewski and Wieczorkowskki (1999) removed the moment condition. (For a finer analysis of the convergence of $H_{m,n}$ to the corresponding population entropy the reader is referred to Song (2000)). Taufer (2002) considers $H_{m,n}$, as well as the entropy estimator of Van Es (1992), and alternative transformations of the original observations, with a view towards maximizing the power of the resulting test. As in the case of the previous statistic, his recommendation is to employ the transformed observations $U_j, 1 \leq j \leq n-1$, and the corresponding entropy estimator $H_{m,n-1}$. Taufer (2002) notes that the test which rejects H_0 for small values of $H_{m,n-1}$ is consistent.

2.5. The statistic of Cox and Oakes

Summarizing the simulation results of some 15 test statistics for exponentiality, Ascher (1990) concludes that if nothing a priori is known about the alternative distribution, the test of Cox and Oakes (1984), which rejects the null hypothesis for both small and large values of

$$CO_n = n + \sum_{j=1}^{n} (1 - Y_j) \log Y_j,$$

is the 'best'. To derive the asymptotic null distribution of CO_n assume (without loss of generality) that $\mathrm{E}(X) = 1$ and notice that

$$\frac{1}{\sqrt{n}} CO_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left[1 + (1 - Y_j) \log X_j\right].$$

Putting $g(\theta) = (1 - X/\theta) \log X$, a Taylor expansion of $g(\hat{\theta}_n)$ around $\theta = 1$ yields

$$\frac{1}{\sqrt{n}} CO_n \approx \frac{1}{\sqrt{n}} \sum_{j=1}^{n} W_j, \tag{2.2}$$

where $W_j = 1 + (1 - X_j) \log X_j + (1 - \gamma)(X_j - 1)$, $\gamma = 0.5772\ldots$ denotes Euler's constant, and $X \approx Y$ means that the random variables X and Y have the same asymptotic distribution. Since $\mathrm{E}(W_1) = 0$ and $\mathrm{E}(W_1^2) = \pi^2/6$, the Central Limit Theorem and (2.2) imply that the limit null distribution of $(6/n)^{1/2}(CO_n/\pi)$ is standard normal. The test based on CO_n is consistent against finite-mean distributions (say $\mu = 1$) with $\mathrm{E}[X \log X - \log X] \neq 1$, provided that the latter expectation exists.

2.6. Tests based on a characterization via the mean residual life function

Under the assumption $0 < \mu < \infty$, the distribution of X is exponential if, and only if $\mathrm{E}(X - t | X > t) = \mu$ for each $t > 0$. Since this condition is equivalent to $\mathrm{E}\,[\min(X, t)] = \mu \cdot F(t)$ for each $t > 0$, Baringhaus and Henze (2000) suggested the Kolmogorov-Smirnov type statistic

$$\overline{KS}_n = \sqrt{n}\ \sup_{t\geq 0}\left|\frac{1}{n}\sum_{j=1}^{n}\min(Y_j,t)\ -\ \frac{1}{n}\sum_{j=1}^{n}\ \mathbf{1}\{Y_j\leq t\}\right|$$
$$= \sqrt{n}\ \max(KS_n^{+}, KS_n^{-}),$$

where

$$KS_n^{+} = \max_{j=0,1,\dots,n-1}\left[\frac{1}{n}(Y_{(1)}+\cdots+Y_{(j)})\ +\ Y_{(j+1)}\left(1-\frac{j}{n}\right)\ -\ \frac{j}{n}\right],$$
$$KS_n^{-} = \max_{j=0,1,\dots,n-1}\left[\frac{j}{n}\ -\ \frac{1}{n}(Y_{(1)}+\cdots+Y_{(j)})\ -\ Y_{(j)}\left(1-\frac{j}{n}\right)\right],$$

and the Cramér-von Mises type statistic

$$\overline{CM}_n = n\int_0^{\infty}\left(\frac{1}{n}\sum_{j=1}^{n}\min(Y_j,t)\ -\ \frac{1}{n}\sum_{j=1}^{n}\ \mathbf{1}\{Y_j\leq t\}\right)^2\exp(-t)dt$$
$$= \frac{1}{n}\sum_{j,k=1}^{n}\left[2-3e^{-\min(Y_j,Y_k)}-2\min(Y_j,Y_k)(e^{-Y_j}+e^{-Y_k})+2e^{-\max(Y_j,Y_k)}\right].$$

Rejection of H_0 is for large values of $\overline{KS}_n$ or $\overline{CM}_n$. The asymptotic null distributions of $\overline{KS}_n$ and $\overline{CM}_n$ are the same as the limit distributions of the classical Kolmogorov-Smirnov and Cramér-von Mises statistics when testing the simple hypothesis of uniformity on the unit interval (see Theorem 1 of Baringhaus and Henze (2000)).

If $0 < \mu = \mathrm{E}(X) < \infty$, then

$$\frac{\overline{KS}_n}{\sqrt{n}} \to \sup_{z\geq 0}\left|\frac{1}{\mu}\mathrm{E}(\min(X,z) - P(X\leq z)\right|$$

and

$$\frac{\overline{CM}_n}{n} \to \int_0^{\infty}\left[\mathrm{E}\left(\min\left(\frac{X}{\mu},z\right)\right) - P\left(\frac{X}{\mu}\leq z\right)\right]^2 e^{-z}dz$$

in probability as $n\to\infty$ (Theorem 2 of Baringhaus and Henze (2000)). If $\mathrm{E}(X)=\infty$, then $\overline{KS}_n/\sqrt{n}\to 1$ and $\overline{CM}_n/n\to 1$ in probability. It follows that the tests based on $\overline{KS}_n$ or $\overline{CM}_n$ are consistent against each fixed alternative distribution having positive, possibly infinite, mean.

2.7. Test statistics derived from the empirical Laplace transform

In these tests, the Laplace transform $\psi(t) = (1+t)^{-1}$ of the unit exponential distribution is estimated by its empirical counterpart $\psi_n(t) = n^{-1}\sum_{j=1}^{n}\exp(-tY_j)$.

2.7.1. The test of Baringhaus and Henze (1991)

This approach uses the fact that ψ satisfies the differential equation $(1+t)\psi'(t)+\psi(t)=0$, $t\in\mathbb{R}$. Consequently, choosing a constant $a>0$ and rejecting H_0 for large values of

$$BH_n = n\int_0^\infty \left[(1+t)\psi_n'(t)+\psi_n(t)\right]^2 \exp(-at)dt$$
$$= \frac{1}{n}\sum_{j,k=1}^{n}\left[\frac{(1-Y_j)(1-Y_k)}{Y_j+Y_k+a} - \frac{Y_j+Y_k}{(Y_j+Y_k+a)^2} + \frac{2Y_jY_k}{(Y_j+Y_k+a)^2} + \frac{2Y_jY_k}{(Y_j+Y_k+a)^3}\right]$$

should give a reasonable test of H_0. Baringhaus and Henze (1991) proved that BH_n has a nondegenerate limiting null distribution. Since

$$\frac{BH_n}{n} \to \int_{-\infty}^{\infty}\left((1+t)\psi_0'(t)+\psi_0(t)\right)^2 \exp(-at)dt$$

in probability, where $\psi_0(t) = \mathrm{E}[\exp(-tX/\mu)]$ is the Laplace transform of X/μ, the test is consistent against any distribution with (positive) finite mean μ. As $a \to \infty$, the test statistic approaches the square of the first component of the smooth test for exponentiality, which is

$$\hat{U}_{n2} = \frac{\sqrt{n}}{2}\left(\frac{1}{n}\sum_{j=1}^{n} Y_j^2 - 2\right) \tag{2.3}$$

(see Baringhaus et al. (2000)).

2.7.2. The test of Henze (1993)

In contrast to BH_n, the test of Henze (1993) employs a 'more direct' L^2-distance type statistic between ψ_n and ψ and rejects the null hypothesis for large values of

$$HE_n = n\int_0^\infty \left(\psi_n(t) - \frac{1}{1+t}\right)^2 \exp(-at)dt$$
$$= \frac{1}{n}\sum_{j,k=1}^{n}\frac{1}{Y_j+Y_k+a} - 2\sum_{j=1}^{n}\exp(Y_j+a)E_1(Y_j+a) + n(1-a\exp(a)E_1(a)),$$

where $E_1(z) = \int_z^\infty [\exp(-t)/t]dt$ is the exponential integral and $a > 0$ is a constant.

Under H_0 the statistic HE_n has a nondegenerate limiting distribution. Because of the stochastic convergence

$$\frac{HE_n}{n} \to \int_0^\infty \left|\psi_0(t) - \frac{1}{1+t}\right|^2 \exp(-at)dt$$

if $0 < \mathrm{E}(X) < \infty$ and $HE_n/n \to \int_0^\infty |1-\frac{1}{1+t}|^2 \exp(-at)dt$ if $\mathrm{E}(X) = \infty$, the test is consistent against any fixed alternative distribution not degenerate at zero. Just as BH_n, also HE_n is connected with the first nonzero component of the smooth test for exponentiality as $a \to \infty$ (see Baringhaus et al. (2000)).

2.7.3. The test of Henze and Meintanis (2002a)

Apart from the tests based on BH_n and HE_n, we cross-reference simulation results from Henze and Meintanis (2002a) in which the generalized version

$$L_n = n\int_0^\infty [(1+t)\psi_n(t)-1]^2 w(t)\exp(-at)dt$$

of HE_n is considered (notice that the choice $w(t) = 1/(1+t)^2$ yields the statistic HE_n). The weight function w is assumed to satisfy $w(t) = O(t^k)$ as $t \to \infty$, for some integer k. Henze and Meintanis (2002a) derive the limit distribution of L_n under H_0 and under contiguous alternatives to exponentiality. Since

$$\frac{L_n}{n} \to \int_0^\infty ((1+t)\psi_0(t)-1)^2 w(t)\exp(-at)dt$$

if $0 < \mathrm{E}(X) < \infty$ and $L_n/n \to \int_0^\infty t^2 w(t)\exp(-at)dt$ if $\mathrm{E}(X) = \infty$ (see Theorem 2.7 of Henze and Meintanis (2002a)), the test based on L_n is consistent against any fixed alternative distribution not degenerate at zero provided that $w(t) > 0$ for each t.

If the weight function w satisfies $w(t) = Ct^m + o(t^m)$ as $t \to 0$ for some $C > 0$ and $m \geq 0$, then $\lim_{a\to\infty} a^{m+5} L_n = C\Gamma(m+5)\hat{U}_{n2}^2$, where $\hat{U}_{n2}$ is given in (2.3).

2.8. Test statistics derived from the empirical characteristic function

In these tests, the characteristic function (CF) $\phi(t) = C(t) + iS(t)$ of X is estimated by the empirical CF (ECF)

$$\phi_n(t) = \frac{1}{n}\sum_{j=1}^n \exp(itX_j) = C_n(t) + iS_n(t).$$

The ECF has a long history as a tool for statistical inference. For goodness-of-fit problems in particular, the methods date back to Heathcote (1972), Koutrouvelis and Kellermeier (1981), S. Csörgő and Heathcote (1982), S. Csörgő (1985), and S. Csörgő and Heathcote (1987). More recent work includes, among others, Henze et al. (2003), Gürtler and Henze (2000), Koutrouvellis and Meintanis (1999) and Kankainen and Usakakov (1998). A large part of the literature on the ECF is covered in Ushakov (1999).

In this paper we compare the test of Epps and Pulley (1986) and a new test which is based on a characterization of exponentiality involving the CF. The new statistic is similar in spirit to the statistic of Henze and Meintanis (2002b) which was motivated by a different characterization of exponentiality via the CF. For details on both characterizations the reader is referred to Meintanis and Iliopoulos (2003). We cross-reference simulation results from Henze and Meintanis (2002b), thereby including all three tests for exponentiality utilizing the empirical CF.

2.8.1. The test of Epps and Pulley (1986)
Notice that, if the distribution of X is $Exp(\theta)$, the ECF $\phi_n(t)$ of $X_1, \ldots, X_n$ should be close to the parametric estimator $\varphi(t;\theta) = 1/(1 - i\bar{X}_n t)$ of the CF of $Exp(\theta)$. The normalized Epps-Pulley test statistic is

$$EP_n = (48n)^{1/2} \int_{-\infty}^{\infty} \left(\phi_n(t) - \frac{1}{1 - i\bar{X}_n t} \right) \frac{\bar{X}_n}{2\pi(1 + i\bar{X}_n t)} dt$$
$$= (48n)^{1/2} \left[\frac{1}{n} \sum_{j=1}^{n} \exp(-Y_j) - \frac{1}{2} \right].$$

Epps and Pulley (1986) show that the limit null distribution of EP_n is standard normal, and that the test which rejects the hypothesis H_0 for large values of $|EP_n|$ is consistent against each alternative distribution with monotone hazard rate, provided that F is absolutely continuous, $F(0) = 0$, and $0 < \mu < \infty$. Curiously enough, although motivated via the empirical characteristic function, EP_n is essentially an estimator of the value of the *Laplace* transform $\mathrm{E}[\exp(-tX/\mu)]$ of the (rescaled) underlying distribution, evaluated at $t = 1$, since

$$\frac{EP_n}{(48n)^{1/2}} + \frac{1}{2} = \frac{1}{n} \sum_{j=1}^{n} \exp(-Y_j) \rightarrow \mathrm{E}[\exp(-X/\mu)]$$

almost surely as $n \to \infty$.

2.8.2. The test of Henze and Meintanis (2002b)

This procedure is based on the fact that the distribution of X is $Exp(\theta)$ if, and only if, the CF of X satisfies the equation $S(t) = \theta t C(t)$, $t \in \mathbb{R}$ (Meintanis and Iliopoulos (2003)). Writing $c_n(\cdot)$ and $s_n(\cdot)$ for the real and the imaginary part, respectively, of the ECF $\varphi_n(t) = n^{-1} \sum_{j=1}^{n} \exp(itY_j)$ of the scaled data $Y_1, \ldots, Y_n$, the statistic of Henze and Meintanis (2002b) is

$$W_n = n \int_{-\infty}^{\infty} (s_n(t) - tc_n(t))^2 w(t) dt, \tag{2.4}$$

where $w(\cdot)$ denotes a nonnegative weight function satisfying $\int_0^\infty t^2 w(t) dt < \infty$.

Henze and Meintanis (2002b) prove that W_n has a nondegenerate limit null distribution. Under a fixed alternative distribution satisfying $0 < \mathrm{E}(X) \leq \infty$, W_n/n has a positive stochastic limit as $n \to \infty$ provided that $w(t) > 0$ for each t. Consequently, a test for exponentiality that rejects H_0 for large values of W_n is universally consistent.

Closed-form expressions for W_n in terms of sums arise for the weight functions $w_1(t) = \exp(-at)$ and $w_2(t) = \exp(-at^2)$, $a > 0$, in (2.4). Writing $W_n^{(1)}$ and $W_n^{(2)}$ for the resulting statistics, we have

$$W_n^{(1)} = \frac{a}{2n} \sum_{j,k=1}^{n} \left[\frac{1}{a^2 + (Y_j - Y_k)^2} - \frac{1}{a^2 + (Y_j + Y_k)^2} - \frac{4(Y_j + Y_k)}{(a^2 + (Y_j + Y_k)^2)^2} + \frac{2a^2 - 6(Y_j - Y_k)^2}{(a^2 + (Y_j - Y_k)^2)^3} + \frac{2a^2 - 6(Y_j + Y_k)^2}{(a^2 + (Y_j + Y_k)^2)^3} \right]$$

and

$$W_n^{(2)} = \frac{\sqrt{\pi}}{4n\sqrt{a}} \sum_{j,k=1}^{n} \left[\left(1 + \frac{2a - (Y_j - Y_k)^2}{4a^2} \right) \exp\left(-\frac{(Y_j - Y_k)^2}{4a} \right) + \left(\frac{2a - (Y_j + Y_k)^2}{4a^2} - \frac{Y_j + Y_k}{a} - 1 \right) \exp\left(-\frac{(Y_j + Y_k)^2}{4a} \right) \right].$$

2.8.3. A new test for exponentiality based on the ECF

Meintanis and Iliopoulos (2003) proved that the distribution of a random variable X is exponential if, and only if, its CF satisfies the equation

$$|\phi(t)|^2 = C(t), \qquad t \in \mathbb{R},$$

where $|\phi(t)|^2 = C^2(t) + S^2(t)$ is the squared modulus of ϕ. Consequently, a reasonable test for exponentiality may be based on the statistic

$$T_n = n \int_0^\infty \left(|\varphi_n(t)|^2 - c_n(t)\right)^2 w(t)dt,$$

where $w(\cdot)$ is a nonnegative weight function. Henze and Meintanis (2002c) proved that under the condition $\int_0^\infty t^4 w(t)dt < \infty$, the limit null distribution of T_n is that of $\int_{-\infty}^\infty \mathscr{W}^2(t)w(t)dt$, where $\mathscr{W}$ is some zero mean Gaussian element in the Hilbert space $L^2(\mathbb{R}, \mathscr{B}, w(t)dt)$. Under a fixed alternative X with $\mathrm{E}(X) = 1$ (this assumption is made without loss of generality because of scale invariance) and $\mathrm{E}(X^2) < \infty$, we have

$$\sqrt{n}\left(\frac{T_n}{n} - \Delta\right) \xrightarrow{\mathscr{D}} N(0, \sigma_0^2), \text{ as } n \to \infty,$$

where

$$\Delta = \int_{-\infty}^\infty [|\phi(t)|^2 - C(t)]^2 w(t)dt$$

and σ_0^2 is given in Henze and Meintanis (2002c). Henze and Meintanis (2002c) also obtained the limit distribution of T_n under contiguous alternatives to H_0.

As for W_n, closed-form expressions for T_n in terms of sums arise for the weight functions $w_1(t) = \exp(-at)$ and $w_2(t) = \exp(-at^2)$, $a > 0$. The resulting statistics are denoted by $T_{n,a}^{(1)}$ and $T_{n,a}^{(2)}$, respectively. Putting $Y_{jk-} = Y_j - Y_k$ and $Y_{jk+} = Y_j + Y_k$, straightforward algebra yields

$$\begin{aligned} T_{n,a}^{(1)} = & \frac{a}{n} \sum_{j,k=1}^n \left[\frac{1}{a^2 + Y_{jk-}^2} + \frac{1}{a^2 + Y_{jk+}^2}\right] \\ & - \frac{2a}{n^2} \sum_{j,k=1}^n \sum_{l=1}^n \left[\frac{1}{a^2 + [Y_{jk-} - Y_l]^2} + \frac{1}{a^2 + [Y_{jk-} + Y_l]^2}\right] \\ & + \frac{a}{n^3} \sum_{j,k=1}^n \sum_{l,m=1}^n \left[\frac{1}{a^2 + [Y_{jk-} - Y_{lm-}]^2} + \frac{1}{a^2 + [Y_{jk-} + Y_{lm-}]^2}\right] \end{aligned}$$

and

$$\begin{aligned} T_{n,a}^{(2)} = & \frac{1}{2n}\sqrt{\frac{\pi}{a}} \sum_{j,k=1}^n \left[\exp\left(-\frac{Y_{jk-}^2}{4a}\right) + \exp\left(-\frac{Y_{jk+}^2}{4a}\right)\right] \\ & - \frac{1}{n^2}\sqrt{\frac{\pi}{a}} \sum_{j,k=1}^n \sum_{l=1}^n \left[\exp\left(-\frac{[Y_{jk-} - Y_l]^2}{4a}\right) + \exp\left(-\frac{[Y_{jk-} + Y_l]^2}{4a}\right)\right] \\ & + \frac{1}{2n^3}\sqrt{\frac{\pi}{a}} \sum_{j,k=1}^n \sum_{l,m=1}^n \left[\exp\left(-\frac{[Y_{jk-} - Y_{lm-}]^2}{4a}\right) + \exp\left(-\frac{[Y_{jk-} + Y_{lm-}]^2}{4a}\right)\right]. \end{aligned}$$

Both $T_{n,a}^{(1)}$ and $T_{n,a}^{(2)}$ are related to the time-honored first component $\hat{U}_{n2}$ of the smooth test for exponentiality, since $\lim_{a\to\infty} a^5\ T_{n,a}^{(1)} = 48\hat{U}_{n2}^2$ and $\lim_{a\to\infty} a^{5/2}$ $T_{n,a}^{(2)} = 3\sqrt{\pi}\hat{U}_{n2}^2/4$. A similar observation was made by Henze and Meintanis (2002b) for the statistic W_n, and by Baringhaus et al. (2000) for other weighted integral test statistics in different goodness-of-fit testing problems.

3. Simulations

This section presents the results of a Monte Carlo study conducted to assess the power of the tests under discussion. All calculations were done using double precision arithmetic in FORTRAN and routines from the IMSL library, whenever available. Empirical critical values of size α for all tests were obtained from 50 000 replications. For the new tests $T_{n,a}^{(1)}$ and $T_{n,a}^{(2)}$, these are given in Table 1. With these critical values, the power of the tests was simulated based on samples of size $n = 20$ and $n = 50$ from the following distributions:

- the Weibull distribution with density $\theta x^{\theta-1}\exp(-x^\theta)$, denoted by $W(\theta)$,
- the Gamma distribution with density $\Gamma(\theta)^{-1}x^{\theta-1}\exp(-x)$, denoted by $\Gamma(\theta)$,
- the lognormal law $LN(\theta)$ with density $(\theta x)^{-1}(2\pi)^{-1/2}\exp(-(\log x)^2/(2\theta^2))$,
- the half-normal HN distribution with density $(2/\pi)^{1/2}\exp(-x^2/2)$,
- the uniform distribution U with density 1, $0 \le x \le 1$,
- the modified extreme value $EV(\theta)$, with distribution function $1 - \exp(\theta^{-1}(1 - e^x))$,
- the linear increasing failure rate law $LF(\theta)$ with density $(1 + \theta x)\exp(-x - \theta x^2/2)$,
- Dhillon's (1981) law $DL(\theta)$, with distribution function $1 - \exp(-(\log(x + 1))^{\theta+1})$,
- Chen's (2000) distribution $CH(\theta)$, with distribution function $1 - \exp(2(1- e^{x^\theta}))$.

These distributions comprise widely used alternatives to the exponential model and include densities f with decreasing hazard rates (DHR) $f(x)/[1 - F(x)]$, increasing hazard rates (IHR) as well as models with non-monotone hazard functions.

Table 1. Critical points for $T_{n,a}^{(1)}$ (top) and $T_{n,a}^{(2)}$ (bottom)

a	$\alpha = 0.05$		$\alpha = 0.1$	
	$n = 20$	$n = 50$	$n = 20$	$n = 50$
0.50	3.21	2.61	2.28	1.93
	1.10	0.931	0.711	0.655
1.0	0.999	0.826	0.667	0.591
	0.443	0.401	0.301	0.285
1.5	0.422	0.359	0.280	0.256
	0.244	0.226	0.171	0.162
2.5	0.113	0.103	0.078	0.074
	0.104	0.104	0.076	0.074
5.0	0.012	0.012	0.0089	0.0089
	0.028	0.030	0.021	0.022

Table 2. Percentage of Monte Carlo samples declared significant by various tests for the exponential distribution ($n = 20$ left, $n = 50$ right)

altern.	EP	$\overline{KS}$	$\overline{CM}$	ω^2	KS	KL	S	CO	EP	$\overline{KS}$	$\overline{CM}$	ω^2	KS	KL	S	CO
$W(0.8)$	24	14	22	20	17	28	24	28	48	35	46	44	38	53	48	56
$W(1.4)$	36	35	35	34	28	29	35	37	80	71	77	75	64	72	79	82
$\Gamma(0.4)$	76	62	75	76	71	88	76	91	99	97	99	99	98	*	98	*
$\Gamma(1.0)$	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
$\Gamma(2.0)$	48	46	47	47	40	44	46	54	91	86	90	90	83	93	90	96
$LN(0.8)$	25	28	27	33	30	35	24	33	45	62	60	76	71	92	47	66
$LN(1.5)$	67	55	66	62	58	66	67	60	95	92	95	94	91	94	95	92
HN	21	24	22	21	18	16	21	19	54	50	53	48	39	37	54	45
U	66	72	70	66	52	61	70	50	98	99	99	98	93	97	99	91
$CH(0.5)$	63	47	61	61	56	77	63	80	94	90	94	95	92	99	94	99
$CH(1.0)$	15	18	16	14	13	11	15	13	38	36	37	32	26	23	38	30
$CH(1.5)$	84	79	83	79	67	76	84	81	*	*	*	*	98	*	*	*
$LF(2.0)$	28	32	30	28	24	23	29	25	69	65	69	64	53	54	69	60
$LF(4.0)$	42	44	43	41	34	34	42	37	87	82	87	83	72	75	87	80
$EV(0.5)$	15	18	16	14	13	11	15	13	38	36	37	32	26	23	38	30
$EV(1.5)$	45	48	47	43	35	37	46	37	90	88	90	86	75	79	90	78
$DL(1.0)$	20	22	21	23	20	21	19	25	39	43	44	52	46	66	39	55
$DL(1.5)$	64	62	63	65	56	63	62	72	97	96	97	98	95	99	97	99

Table 3. Percentage of Monte Carlo samples declared significant by the entropy test based on $H_{m,n-1}$ for $m = 4, 5, 6, 8$ ($n = 20$) and $m = 10, 12, 14, 16, 20$ ($n = 50$)

altern.	$n = 20$				$n = 50$				
	4	5	6	8	10	12	14	16	20
$W(0.8)$	11	10	10	9	21	19	17	15	13
$W(1.4)$	20	20	20	19	53	53	52	51	47
$\Gamma(0.4)$	51	45	38	27	90	85	79	72	55
$\Gamma(1.0)$	5	5	5	5	5	5	5	5	5
$\Gamma(2.0)$	33	35	36	36	81	82	82	82	80
$LN(0.8)$	47	51	53	56	90	96	96	96	86
$LN(1.5)$	55	55	54	51	90	90	90	89	86
HN	8	8	8	7	20	19	17	16	12
U	31	27	23	16	86	80	72	63	44
$CH(0.5)$	35	30	25	17	74	67	59	51	36
$CH(1.0)$	6	6	5	5	11	10	9	8	7
$CH(1.5)$	47	47	46	40	96	96	94	93	87
$LF(2.0)$	12	12	11	10	30	29	27	24	20
$LF(4.0)$	18	18	17	16	50	48	46	43	35
$EV(0.5)$	6	6	5	5	11	10	9	8	7
$EV(1.5)$	17	16	15	12	52	48	44	39	29
$DL(1.0)$	26	28	29	30	63	65	66	67	67
$DL(1.5)$	51	54	55	55	96	97	97	97	96

For the nominal level 5%, Tables 2-5 show power estimates of the tests under discussion. The entries are the percentages of 10 000 Monte Carlo samples that resulted in rejection of H_0, rounded to the nearest integer. An asterisk denotes power 100%.

Table 4. Percentage of Monte Carlo samples of size $n = 20$ declared significant by the tests based on the empirical Laplace transform (left: *BH* top, *HE* bottom) and the new tests based on the empirical CF (right: $T_{n,a}^{(1)}$ top, $T_{n,a}^{(2)}$ bottom)

altern.	0.5	1.0	1.5	2.5	5.0	0.5	1.0	1.5	2.5	5.0
$W(0.8)$	26	25	24	24	24	1	1	1	4	15
	26	25	24	24	24	1	3	5	10	19
$W(1.4)$	36	37	37	36	32	40	44	45	45	40
	37	38	37	37	36	45	45	44	42	36
$\Gamma(0.4)$	87	83	80	77	72	0	2	11	33	57
	86	82	79	79	76	10	26	37	50	60
$\Gamma(1.0)$	5	5	5	5	5	5	5	5	5	5
	5	5	5	5	5	5	5	5	5	5
$\Gamma(2.0)$	53	53	51	48	43	55	57	56	55	48
	54	53	51	48	42	55	55	52	50	44
$LN(0.8)$	37	33	29	26	22	49	39	34	27	22
	36	32	29	29	26	31	26	24	22	20
$LN(1.5)$	63	65	66	67	68	0	0	2	18	58
	63	65	66	66	67	1	10	21	38	62
HN	18	20	21	21	19	23	28	31	33	29
	19	20	21	21	21	31	33	33	31	25
U	50	58	61	65	66	69	78	82	86	84
	51	58	62	62	65	83	86	87	85	81
$CH(0.5)$	75	70	67	63	59	0	1	6	23	43
	73	69	66	63	59	4	18	27	37	48
$CH(1.0)$	13	14	15	14	13	16	21	23	25	22
	13	14	15	14	14	23	25	25	23	18
$CH(1.5)$	78	82	83	84	82	81	87	89	91	90
	80	82	83	84	82	90	91	91	91	87
$LF(2.0)$	25	28	25	28	27	30	36	39	42	37
	26	28	28	28	28	40	42	41	39	33
$LF(4.0)$	37	40	41	42	40	43	50	54	56	52
	38	40	42	42	42	54	57	56	54	47
$EV(0.5)$	13	14	15	14	13	16	21	23	25	22
	13	14	15	15	14	23	25	25	23	18
$EV(1.5)$	36	41	43	44	43	43	53	58	63	59
	37	41	43	43	44	60	64	63	61	54
$DL(1.0)$	25	24	23	21	17	34	30	28	25	20
	26	24	22	20	18	27	24	23	21	18
$DL(1.5)$	72	70	68	64	57	74	73	71	67	60
	73	70	68	64	57	69	66	64	61	56

The main conclusions that can be drawn from the simulation results are the following:

1. For $n = 20$, the most powerful among the tests in Tables 2 and 3 are the tests based on EP_n, $\overline{KS}_n$, $\overline{CM}_n$, S_n and CO_n. For $n = 50$, the same holds true, with the exception of the test based on $\overline{KS}_n$ which performs less favorably, and it is replaced by the classical Cramér-von Mises test.
2. From the figures in Tables 4 and 5 it is evident that the power of the tests based on BH_n and HE_n, as well as that of the new tests based on $T_{n,a}^{(1)}$ and $T_{n,a}^{(2)}$, depends on the value of the weight parameter a, for some alternatives more drastically than for other alternatives. The fact that no

Table 5. Percentage of Monte Carlo samples of size $n = 50$ declared significant by the tests based on the empirical Laplace transform (left: *BH* top, *HE* bottom) and the new tests based on the empirical CF (right: $T_{n,a}^{(1)}$ top, $T_{n,a}^{(2)}$ bottom)

altern.	0.5	1.0	1.5	2.5	5.0	0.5	1.0	1.5	2.5	5.0
$W(0.8)$	53	51	50	48	47	6	12	17	25	35
	52	51	50	48	48	16	22	27	31	38
$W(1.4)$	80	80	81	79	76	75	80	81	81	78
	81	81	81	79	76	81	82	81	80	76
$\Gamma(0.4)$	*	99	99	99	97	82	87	90	92	94
	*	99	99	99	97	87	91	92	93	94
$\Gamma(1.0)$	5	5	5	5	5	5	5	5	5	5
	5	5	5	5	5	5	5	5	5	5
$\Gamma(2.0)$	95	94	93	91	86	92	93	92	90	84
	95	94	93	91	87	91	89	88	85	82
$LN(0.8)$	81	67	58	47	36	89	77	66	49	31
	74	62	55	46	38	60	46	40	32	27
$LN(1.5)$	93	94	95	95	95	16	36	54	74	91
	93	94	95	95	95	47	65	74	83	92
HN	44	49	52	53	53	45	55	60	64	63
	45	50	52	53	53	60	64	65	64	61
U	93	96	97	98	99	98	99	*	*	*
	92	96	97	98	99	*	*	*	*	*
$CH(0.5)$	98	97	96	95	92	65	74	79	83	85
	98	97	96	94	92	76	80	83	84	85
$CH(1.0)$	29	33	35	37	37	30	39	44	48	48
	29	34	35	37	37	43	48	50	49	46
$CH(1.5)$	*	*	*	*	*	*	*	*	*	*
	*	*	*	*	*	*	*	*	*	*
$LF(2.0)$	60	65	68	69	68	61	70	74	77	76
	61	66	68	69	69	74	78	79	78	75
$LF(4.0)$	80	84	86	87	87	80	87	90	91	91
	81	85	86	87	87	89	91	92	92	90
$EV(0.5)$	29	33	35	37	37	30	39	44	48	48
	29	34	35	37	37	43	48	50	49	46
$EV(1.5)$	79	85	87	89	91	82	90	93	95	96
	80	85	87	89	91	93	95	96	96	95
$DL(1.0)$	60	52	47	40	33	67	60	54	43	30
	58	51	46	40	34	50	42	37	32	27
$DL(1.5)$	99	99	98	97	94	99	98	98	96	92
	99	99	98	97	94	97	96	95	93	89

prior knowledge about possible deviations from exponentiality is assumed, calls for a compromise solution, and it seems that the tests corresponding to $a = 1.5$ or $a = 2.5$ satisfactorily fulfill this objective. Also, since the BH_n and the HE_n tests exhibit a very similar performance, we hereafter refer only to the BH_n test, and denote this test by $BH_{n,a}$.

3. The performance of the best tests in Tables 2 and 3, the test of Baringhaus and Henze (1991), and the new tests based on $T_{n,1.5}^{(j)}$ and $T_{n,2.5}^{(j)}$, $j = 1, 2$, was compared by ranking, for each alternative distribution. When $n = 20$, the $T_{n,2.5}^{(1)}$ test dominates, and despite the fact that the new tests perform poorly under DHR alternatives, they clearly rank at the top, followed by

the $BH_{n,1.5}$, the CO_n, and the test based on $\overline{CM}_n$, in this order. For $n = 50$, the differences in power are less pronounced, and with the exception of the somewhat less powerful classical Cramér-von Mises test, all tests perform comparably. However, the tests based on $T^{(1)}_{n,1.5}$, $T^{(2)}_{n,1.5}$, and $T^{(1)}_{n,2.5}$, rank at the top, followed by the $\overline{CM}_n$, the $BH_{n,1.5}$, and the test of Epps and Pulley.

We also compared, when possible, the new tests based on $T^{(j)}_{n,1.5}$ and $T^{(j)}_{n,2.5}$, $j = 1, 2$ with the tests of Henze and Meintanis (2002a, 2002b). The latter tests are more powerful in cases of DHR or non-monotone hazard rate alternatives, while the new tests are preferable for alternatives with IHR. In conclusion, the new tests are serious competitors, and perhaps should be employed, in the absence of any information regarding the type of deviation from exponentiality. When such information exists and indicates an IHR alternative, the new tests are powerful, but then they should also be compared with tests that are designed to specifically guard against such deviations. However, when a deviation towards DHR alternatives is more likely, the new tests should not be employed. With an appropriate choice of the weight parameter, the tests of Baringhaus and Henze (1991) and Henze (1993) are among the most powerful tests. Also, between the two methods utilizing the mean residual life function, the Cramér-von Mises type test exhibited a very competitive performance, and among the more classical procedures the test of Cox and Oakes.

The test of Epps and Pulley and the test based on spacings were competitive only for $n = 50$, while the methods involving entropy did not compete well for either sample size. Among the methods involving the EDF (or the integrated EDF), the Kolmogorov-Smirnov test is the least powerful. The test of Klar (2001) competes well with the classical Cramér-von Mises test but, at least in the cases of alternative distributions considered here, does not rank at the top.

Acknowledgements Research on this topic started while the first author was visiting the University of Patras. Norbert Henze would like to thank the Department of Engineering Sciences for its hospitality and strong support. The authors are grateful to the referees for many constructive comments.

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